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ASYMPTOTIC ENUMERATION AND A 0-1 LAW FOR m-CLIQUE FREE GRAPHS

BY PH. G. KOLAITIS, H.-J. PRÖMEL AND B. L. ROTHSCHILD

In this note we announce some results about the asymptotic behavior of K_m-free graphs. These are the undirected finite graphs which do not contain a complete graph K_m with m vertices (an m-clique) as a subgraph. It is obvious that every graph which contains a clique of size $l+1$ is not l-colorable, and hence has chromatic number at least $l+1$. Also it is well known that there are K_{l+1}-free graphs of arbitrarily large chromatic number. In contrast to this we show that "almost-all" K_{l+1}-free graphs are l-colorable, for any $l \geq 2$. More precisely, we establish

THEOREM 1. *Let $S_n(l)$ be the number of labeled K_{l+1}-free graphs on $\{1,2,\ldots,n\}$ and let $L_n(l)$ be the number of labeled l-colorable graphs on $\{1,2,\ldots,n\}$. Then for any polynomial $p(n)$ there is a constant C such that for all n*

$$S_n(l) \leq L_n(l)\left(1+\frac{C}{p(n)}\right)$$

and hence

$$\lim_{n\to\infty}\left(\frac{L_n(l)}{S_n(l)}\right) = 1.$$

The special case of the above theorem for $l=2$ and $p(n)=n$ was proved by Erdös, Kleitman, Rothschild [**1976**], who also showed that

$$\lim_{n\to\infty}\left(\frac{\log L_n(l)}{\log S_n(l)}\right) = 1 \quad \text{for any } l \geq 2.$$

In addition to the asymptotic enumeration given by Theorem 1, we derive detailed information about the structure of almost all K_{l+1}-free graphs. We use this to prove that the labeled asymptotic probability of any first-order property on the class $\mathcal{S}(l)$ of all finite K_{l+1}-free graphs is either 0 or 1. C. W. Henson (private communication) obtained the first-order 0-1 law for the class $\mathcal{S}(2)$ from the asymptotic results about K_3-free graphs in Erdös, Kleitman, Rothschild [**1976**]. The classes $\mathcal{S}(l)$ of K_{l+1}-free graphs and $\overline{\mathcal{S}}(l)$ of their complementary graphs occur in the Lachlan-Woodrow [**1980**] characterization of classes of finite undirected graphs having the amalgamation property and closed under induced subgraphs. Together with first-order 0-1 laws already known for other such classes (Fagin [**1976**], Compton [**1984**]) we obtain

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THEOREM 2. *Let $\mathcal{K}$ be any infinite class of finite undirected graphs having the amalgamation property and closed under induced subgraphs and isomorphisms. Then the labeled asymptotic probability of any sentence of first-order logic on $\mathcal{K}$ is either* 0 *or* 1.

The proofs of these results will appear elsewhere. In what follows we give a brief description of some of the main ideas.

1. Asymptotic enumeration of K_{l+1}-free graphs. Let $\mathcal{S}_n(l)$ and $\mathcal{L}_n(l)$ be the classes of K_{l+1}-free graphs and l-colorable graphs on $\{1, 2, \ldots, n\}$ respectively.

We first establish an estimate for the growth rate of $L_n(l) = |\mathcal{L}_n(l)|$, namely

$$\log\left(\frac{L_{n+1}(l)}{L_n(l)}\right) \geq \left(\frac{l-1}{l}\right) n - (l+1)\log n \tag{1}$$

for all sufficiently large n (depending on l). Then to prove Theorem 1 we consider $3l$ subclasses of $\mathcal{S}_n(l)$ and show that they exhaust $\mathcal{S}_n(l) - \mathcal{L}_n(l)$. For each of these classes we estimate its size relative to $\mathcal{S}_{n-1}(l)$. We combine these estimates with (1) and induction on n to prove that each class has fewer than $L_n(l) \cdot C/3lp(n)$ elements, and thus is negligible relative to $\mathcal{L}_n(l)$.

A similar method of proof was used by Kleitman, Rothschild [**1975**] and Erdös, Kleitman, Rothschild [**1976**] in the asymptotic enumerations of partial orders and K_3-free graphs respectively. In our case the choice of classes is of course different and, moreover, sufficiently elaborate to yield structural information for those graphs not in any one of the $3l$ negligible classes. In particular, we show the following for almost all K_{l+1}-free graphs:

(a) Each vertex v must be adjacent to a set Q of vertices forming a complete $(l-1)$-partite graph with all parts of size $q(n)$, where $q(n)$ is a certain function such that $\lim_{n\to\infty} q(n) = \infty$.

(b) For each vertex v and each set Q as above, the set R of vertices adjacent to at least one vertex in each part of Q satisfies

$$\frac{n}{l} - \frac{n}{\log q(n)} \leq |R| \leq \frac{n}{l} + n^{1/2}.$$

(c) If $v_1, \ldots, v_l$ are vertices forming a complete graph K_l, then any sets $R_1, \ldots, R_l$ associated with $v_1, \ldots, v_l$ as in (b) are pairwise disjoint. Moreover, any vertex w not in $\bigcup_{i=1}^{l} R_i$ must be adjacent to vertices in exactly $(l-1)$ of the R_i.

2. First order 0-1 laws. Let $\mathcal{K}$ be a class of labeled finite undirected graphs and let φ be a property of graphs expressible by a sentence of first-order logic. The *labeled asymptotic probability* $\mu(\varphi)$ *of* φ *on* $\mathcal{K}$ is given by

$$\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi) \quad \text{(provided this limit exists)},$$

where $\mu_n(\varphi)$ is the fraction of graphs in $\mathcal{K}$ with n vertices, satisfying φ. The *first-order almost sure theory of* $\mathcal{K}$ is defined to be the set

$$T(\mathcal{K}) = \{\varphi \colon \varphi \text{ is a first-order sentence and } \mu(\varphi) = 1 \text{ on } \mathcal{K}\}.$$

Fagin [**1976**] showed that if $\mathcal{G}$ is the class of all finite undirected graphs, then $T(\mathcal{G})$ coincides with the set of first-order sentences which are true on a certain countable graph, known as Rado's graph (Rado [**1964**]). As a result of this, the asymptotic probability $\mu(\varphi)$ of any first-order sentence φ on $\mathcal{G}$ is either 0 or 1. If $\mathcal{E}$ is the class of all finite equivalence relations, then it follows from the work of Compton [**1984**] that a 0-1 law for first-order sentences also holds on $\mathcal{E}$.

Both $\mathcal{G}$ and $\mathcal{E}$ are classes of undirected graphs having the amalgamation property and closed under submodels and isomorphisms. Lachlan and Woodrow [**1980**] gave a complete classification of all families $\mathcal{K}$ of graphs with these properties. First-order 0-1 laws can be established for some other such $\mathcal{K}$'s by applying the results of Compton [**1984**]. Beyond these there are essentially two possibilities for $\mathcal{K}$, namely $\mathcal{S}(l)$ (all K_{l+1}-free graphs) and $\mathcal{E}(m)$, the family of all finite equivalence relations with at most m equivalence classes. The first-order 0-1 law for $\mathcal{E}(m)$ is straightforward.

For each class $\mathcal{S}(l)$ we consider a certain infinite set of first-order axioms which are satisfied by a unique (up to isomorphism) countable undirected graph D_l. This graph is in a sense the analog for l-colorable graphs of Rado's graph. Using the structural properties (a), (b), (c) in §1, we prove that $\mu(\psi) = 1$ on $\mathcal{S}(l)$ for each axiom ψ. It follows that the almost sure theory $T(\mathcal{S}(l))$ coincides with the set of first-order sentences which are true on D_l, and consequently a first-order 0-1 law holds for $\mathcal{S}(l)$. This completes the last case for Theorem 2.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024