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Semigroups of linear operators and applications to partial differential equations, by A. Pazy, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983, viii + 279 pp., $29.80. ISBN 0-387-90845-5

If f is a continuous, real-valued function on $[0, \infty)$ that satisfies $f(0) = 1$ and the semigroup property $f(t+s) = f(t)f(s)$ for $t, s \geqslant 0$, then it is easy to see that there is a real number a such that $f(t) = e^{ta}$ for $t \geqslant 0$ [one can, for example, note that $t \to \ln(f(t))$ is continuous and additive on $[0, \infty)$, and so $\ln(f(t)) = ta$ for $t \geqslant 0$, where $a = \ln(f(1))$]. The number a can also be computed directly from f by the formula

$$(1) \qquad a = \lim_{h \to 0^+} \frac{f(h) - 1}{h} = \frac{d^+}{dt} f(t) \bigg]_{t=0}.$$

This observation is important because it connects f to the initial-value problem

$$(2) \qquad u'(t) = au(t), \quad t \geqslant 0, \qquad u(0) = z.$$

In particular, $u(t) \equiv f(t)z$ is the solution to (2). Conversely, given the initial-value problem (2), there are many ways to construct the solution directly: The three formulas

$$(3) \quad e^{ta} = \sum_{n=0}^{\infty} \frac{t^n}{n!} a^n, \quad e^{ta} = \lim_{n \to \infty} \left(1 + \frac{t}{n} a\right)^n, \quad e^{ta} = \lim_{n \to \infty} \left(1 - \frac{t}{n} a\right)^{-n}$$

are known from elementary calculus to be valid.

Given a basic background in functional analysis, this elementary discussion can be readily extended to various types of initial-value problems. Suppose that H is a Hilbert space with inner product denoted by $\langle \cdot , \cdot \rangle$ and that $\{e_n\}_1^\infty$ is a complete orthonormal sequence in H. Suppose further that $\{\lambda_n\}_1^\infty$ is a decreasing sequence of real numbers, and for each $t \geqslant 0$ and $z \in H$ define

$$(4) \qquad S(t)z = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle z, e_n \rangle e_n.$$

It is easy to see that $S(t)$ is a bounded linear operator on H, with $\|S(t)\| \leqslant e^{\lambda_1 t}$ for all $t \geqslant 0$, and the orthogonality of $\{e_n\}_1^\infty$ implies that the semigroup property $S(t)S(s)z = S(t+s)z$ holds for all $t, s \geqslant 0$ and $z \in H$. A formal calculation shows that

$$\frac{d}{dt} S(t)z \bigg]_{t=0} = \sum_{n=1}^{\infty} \lambda_n \langle z, e_n \rangle e_n,$$

and so [motivated by (1)] we define the operator A on $D(A) \subset H$ by

$$(5) \qquad \begin{cases} Az = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n & \text{for all } z \in D(A), \quad \text{where} \\ D(A) \equiv \left\{ z \in H \colon \sum_{n=1}^{\infty} \lambda_n \langle z, e_n \rangle e_n \text{ exists in } H \right\}. \end{cases}$$

Note that $D(A)$ may be a proper (but dense) subset of H, and if $D(A) \neq H$, then A is unbounded. A further routine calculation shows that if $S(t)z \in D(A)$ for all $t > 0$, then

$$\frac{d}{dt} S(t)z = AS(t)z, \quad t > 0, \qquad S(0)z = z.$$

In particular, given the operator A defined by (5) and given $z \in H$, a solution $u \colon [0, \infty) \to H$ to the initial-value problem

$$(6) \qquad u'(t) = Au(t), \quad t > 0, \qquad u(0) = z,$$

is $u(t) \equiv S(t)z$, where S is defined by (4).

As a specific illustration, suppose that $H = L^2(0, \pi)$, $e_n(\sigma) \equiv \pi^{-1} \sin(n\sigma)$ for $\sigma \in [0, \pi]$, and $\lambda_n = -n^2$. Then for each $z \in L^2(0, \pi)$ and $t \geqslant 0$, the member $S(t)z$ of $L^2(0, \pi)$ is defined on $[0, \pi]$ by

$$[S(t)z](\sigma) = \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\int_0^{\pi} z(\rho) \sin(n\rho)\, d\rho \right] e^{-n^2 t} \sin(n\sigma).$$

Of course, elementary Fourier series techniques show that $y(\sigma, t) \equiv [S(t)z](\sigma)$ is a solution to the heat equation

$$(7) \qquad \begin{cases} y_t(\sigma, t) = y_{\sigma\sigma}(\sigma, t), & t > 0, 0 < \sigma < \pi, \\ y(0, t) = y(\pi, t) = 0, & t > 0, \\ y(\sigma, 0) = z(\sigma), & 0 < \sigma < \pi. \end{cases}$$

As opposed to expressing A in the series representation (5), note that if

$$[u(t)](\sigma) = [S(t)z](\sigma) = y(\sigma, t),$$

then $[u'(t)](\sigma)$ corresponds to $y_t(\sigma, t)$. Comparison of equations (6) and (7) indicates that A has the form

$$[Az](\sigma) = z''(\sigma) \quad \text{for all } z \in D(A),$$

where

$$D(A) \supset \{ z \in L^2(0, \pi) \colon z \text{ is } C^2 \text{ and } z(0) = z(\pi) = 0 \}.$$

A more careful analysis shows that $D(A)$ consists precisely of all $z \in L^2(0, \pi)$ such that z, z' are absolutely continuous, $z'' \in L^2(0, \pi)$, and $z(0) = z(\pi) = 0$. Since the boundary conditions in (7) are absorbed into the domain of A, we see that if u is a solution to (6) [so that $u(t) \in D(A)$ for $t > 0$] and $y(\sigma, t) \equiv [u(t)](\sigma)$, then $y(\cdot, t) \in D(A)$ implies that the boundary conditions in (7) are satisfied. Therefore, the initial-boundary value problem (7) corresponds to an ordinary differential equation (6) in the Hilbert space $L^2(0, \pi)$.

In a general Banach space X a family $T = \{T(t)\colon t \geqslant 0\}$ of bounded linear operators on X is said to be a *semigroup of class* C_0, or simply a C_0 *semigroup*, if

(a) $T(0) = I$ (I the identity operator on X),
(b) $T(t + s) = T(t)T(s)$ for $t, s \geqslant 0$ (the semigroup property),
(c) $t \to T(t)x$ is continuous from $[0, \infty)$ into X for each $x \in X$ (the C_0 property)

If, in addition,

(d) $\|T(t)\| \leqslant 1$ for $t \geqslant 0$ (contraction property).

T is said to be a C_0 *semigroup of contractions*. By appropriate modifications a C_0 semigroup can always be transformed into a C_0 semigroup of contractions, and so in most cases it is sufficient to assume that the semigroup is a contraction.

The (*infinitesimal*) *generator* of T is the operator A on $D(A) \subset X$ defined by

$$(8) \qquad Ax = \lim_{h \to 0_t} \frac{T(h)x - x}{h} \quad \text{for all } x \in D(A),$$

with $D(A)$ being the set of $x \in X$ such that this limit exists [compare with (1)]. Although differentiability is not assumed for T, the semigroup property plus the C_0 property imply that $D(A)$ is always dense in X and T satisfies

$$(9) \qquad \frac{d}{dt}T(t)x = AT(t)x, \quad t \geqslant 0, \qquad T(0)x = x \in D(A)$$

[compare with (2)]. In particular, $T(t)x \in D(A)$ if $x \in D(A)$. Of course, if A is a bounded linear operator then

$$T(t) = e^{tA} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \qquad t \geqslant 0,$$

is the C_0 semigroup whose generator is A. However, the most important applications arise from C_0 semigroups whose generators are unbounded [see, e.g., equation (7)]. In this case the semigroup T is represented in terms of the resolvent of its generator A [see the third formula in (3)].

The first four chapters of Pazy's book deal with the fundamental abstract results and properties of C_0 semigroups and their generators. All of the main topics are presented in a clear and concise manner, and since only an elementary knowledge of functional analysis is assumed, these ideas are accessible to a large number of readers. The Hille-Yosida theorem (characterizing C_0 semigroups and their generators) and several representation theorems are covered. Also included are various results on the differentiability and analyticity of C_0 semigroups as well as results on perturbations and approximations. Another important topic investigated is the nonhomogeneous initial value problem

$$(10) \qquad v'(t) = Av(t) + f(t), \quad t > 0, \qquad v(0) = z,$$

where $f\colon [0, \infty) \to X$ is locally L^1. Applying the standard variation-of-parameters procedure, one can show that if T is the C_0 semigroup generated by A, then any solution v to (10) must be of the form

$$v(t) = T(t)z + \int_0^t T(t-s)f(s)\,ds, \qquad t \geqslant 0. \tag{11}$$

Thus, the function v defined by (11) is called the *mild solution* to (10).

The ideas developed for nonhomogeneous equations are important to the study of nonlinear (or semilinear) equations. Suppose, for example, that f: $[0, b) \times X \to X$ is a function, and consider the equation

$$w'(t) = Aw(t) + f(t, w(t)), \quad 0 < t < b, \qquad w(0) = z. \tag{12}$$

Let $\mathscr{C}$ be the space of all continuous X-valued functions on $[0, b]$ and define the operator $\mathscr{F}$: $\mathscr{C} \to \mathscr{C}$ by

$$[\mathscr{F}\phi](t) = T(t)z + \int_0^t T(t-s)f(s, \phi(s))\,ds, \qquad 0 \leqslant t \leqslant b, \tag{13}$$

for each $\phi \in \mathscr{C}$. Comparison of (10)–(11) with (12)–(13) shows that any solution to (12) must be a fixed point for the mapping $\mathscr{F}$. Results and techniques of this type are studied in Chapter 6. Chapter 5 deals with *linear evolution* equations: that is, abstract linear differential equations that are time dependent (as opposed to time independent in the semigroup case). The techniques are based on semigroup theory, but are considerably more complicated.

The final two chapters of this book study applications of the abstract results to partial differential equations. In order to fully understand these ideas the reader needs a reasonable knowledge of several rather deep results and techniques from the theory of partial differential equations (e.g., Sobolev spaces, Sobolev's imbedding theorem, Gårding's inequality). Under these assumptions these two chapters give an interesting and readable presentation of the way C_0 semigroup theory applies to both linear and nonlinear partial differential equations. It is interesting that if one always assumes that the spacial variable belongs to a compact interval in $\mathbb{R}$ (as opposed to a bounded region Ω in $\mathbb{R}^n$), then most of these results can be developed with only elementary analysis.

Since E. Hille and K. Yosida established the characterization of generators of C_0 semigroups in the 1940s, semigroups of linear operators and its neighboring areas have developed into a beautiful abstract theory. Moreover, the fact that mathematically this abstract theory has many direct and important applications in partial differential equations enhances its importance as a necessary discipline in both functional analysis and differential equations. In my opinion Pazy has done an outstanding job in presenting both the abstract theory and basic applications in a clear and interesting manner. The choice and order of the material, the clarity of the proofs, and the overall presentation make this an excellent place for both researchers and students to learn about C_0 semigroups.

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