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Boundary-value problems with free boundaries for elliptic systems of equations, by V. N. Monakhov, Translations of Mathematical Monographs, Vol. 57, Americal Mathematical Society, Providence, R.I., 1983, xiv + 522 pp, $110.00. ISBN 0-8218-4510-1 (Originally published by "Nauka", Novosibirsk, 1977)

This book breathes new life into a point of view the reviewer had long considered to have expired with his own contribution of 1956. The conceptual starting point comes from the papers of Weinstein (1928–1929), Leray and Weinstein (1934), and Leray (1935–1936). Weinstein developed an earlier suggestion of Weyl to use the continuity method; he proved the unique existence, under some conditions, of a steady two-dimensional incompressible potential flow through a polygonal nozzle with partially free boundaries determined by a constant pressure condition. The result was later improved by Leray and Weinstein, while Leray showed that such problems are, in principle, accessible to fixed-point methods.

All these results imposed (total) curvature conditions on the fixed boundary. The author applies the Weyl-Weinstein procedure to the case of Kirchhoff flow past an obstacle and finds again a unique solution subject to certain curvature conditions. On the other hand, the reviewer (loc. cit.) found analogous results under very different kinds of restrictions; thus it seems likely that the restrictions are intrinsic to the methods and not to the problem. In this sense the continuity method, which is conceptually very appealing, still leaves major questions unsettled.

The author then follows the lead taken by Leray and shows that the Leray-Schauder theorem can be applied to obtain a very general existence result without significant curvature restrictions, but also without a uniqueness or local stability proof. The result is shown to include Kirchhoff flows, Thullen vortex flows, the reentrant jet of Efros, Riabouchinsky and Lavrent′ev flows, flow through a nozzle, and flow in channels with partially unknown walls. Explicit representations for the solutions are given in terms of undetermined parameters.

For many authors this would be a natural stopping point; for this author it became a plateau from which further flights could be initiated. He proceeds at first to show that other kinds of conditions (e.g., gravitational and/or capillary) can be included by an approximation procedure. That brings the reader to about the halfway point in the book. The remainder of the book is devoted to extensions to other flow equations that are governed by quasi-conformal, rather than conformal, mappings. There is a long chapter on mapping theorems for multiply connected domains, by quasi-conformal mappings determined by nonlinear systems; this is followed by a chapter on the Hilbert, and other, boundary value problems and potential and singular operators.

The author now returns to the original conceptual framework and proves an analogous set of results—this time in the context of compressible potential flows. Some restrictions appear, but he does obtain, for example, established general theorems of Lavrent′ev and of Bers as particular cases of broad and embracing theorems. Again, some configurations with gravitation included, and even the case of rotational motion, are considered.

The next two (and final) chapters contain further applications of the ideas to filtration theory and to planar elasticity.

About half of the book is devoted to the presentation of background material. There is a lot of it, it is nicely chosen, and much of it is given in detail with complete proofs. This part of the text could provide the nucleus for a graduate course entitled "Some working tools of modern applied mathematics". At the end of each chapter there is a section in fine print including background discussion, analysis of proofs that were given, discussion of open questions, and proposals for further research. Also here, the presentation is detailed; it is apparent that the author devoted considerable thought to what he wrote.

On reading through the text, the reviewer felt rather overwhelmed by the display of technique and virtuosity. The original papers on the continuity method in relatively simple cases were already technically difficult, and to have developed the idea to the extent that is done here is, by any standard, an impressive achievement. The only feeling of uneasiness arises in connection with the applications currently envisaged. For, in addition to being technically hard and, to a large part, nonconstructive, the methods are basically two-dimensional, or at best axisymmetric. Concomitantly with the writing of the book, other approaches based on direct variational procedures were being developed by Alt, Baiocchi, Caffarelli, Friedman, and others; some of the material in the text can now be obtained alternatively by these procedures, which are not (essentially) limited in dimension and are more accessible to calculation. There is perhaps an analogy with the theorem of Bers noted above; that result was obtained independently by Shiffman by using a much simpler variational method not (essentially) limited to two dimensions. It is at present impossible to predict to what extent the applications in the text will suffer a similar fate. What can be said with certainty is that the book presents with clarity in small space an impressive amount of quality mathematics that will—at the very least—find its own natural realm of applications.

It would be inappropriate to close this review without a note on the Bibliography. It is unfortunate that Levi-Civita, whose representation theorem lies at the base of the entire theory, is not mentioned. Nor is Weyl, who first proposed the continuity method and made significant contributions to it. Neither the book of Cisotti nor the review article by Gilbarg are cited. And the reviewer is impelled to say that his own contribution, mentioned earlier in the review, also does not appear.

These omissions were certainly not deliberate. One gets the impression that the author learned of the approach from the papers of Weinstein and of Leray, took off on his own from there, and had little time or patience to worry about petty historical details. That is, after all, an understandable point of view for an active and creative scientist. Such details are best left to the historians and to the book reviewers, who are usually delighted by the opportunity to fill them in.

R. FINN

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Commutative semigroup rings, by Robert Gilmer, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, IL, xii + 380 pp., 1984, $11.00. ISBN 0-2262-9392-0

Let R be a commutative ring and S a semigroup with respect to an operation $+$, not necessarily commutative. The semigroup ring $R[X; S]$ consists of formal sums $\sum_{i=1}^{n} r_i X^{s_i}$, $r_i \in R$, $s_i \in S$, with addition defined by adding coefficients, and multiplication defined distributively using the rule $X^s X^t = X^{s+t}$. For example, if N is the semigroup of nonnegative integers, then $R[X; N]$ is just the polynomial ring $R[X]$ in a single indeterminate X. Another important example is the semigroup ring $K[X; G]$, where K is a field and G is a finite group. The theory of semigroup rings divides much along the lines of these two examples. If $R[X]$ is taken as the starting point, then the tools and problems come from commutative algebra; if the starting point is $K[X; G]$, then the group G is the primary object of study, and the tools come from group representation theory and constitute a rich mixture of many other areas of mathematics. It should be emphasized that in the case of $K[X; G]$, the main interest is in a nonabelian group G; indeed, a large portion of noncommutative ring theory has been developed specifically in order to deal with this example. (A nice set of lectures on this aspect of the subject, with the ultimate goal of proving a couple of important theorems on finite groups, can be founnd in [**6**].)

To get an idea of the shift in emphasis imposed by restricting to a commutative semigroup S, as is done in this book, consider the question of semisimplicity of $K[X; G]$. A ring is called semisimple if its Jacobson radical is 0. For a commutative ring A, the Jacobson radical $J(A)$ is defined to be the intersection of the maximal ideals of A. A related notion is that of the nilradical $N(A)$, which is the intersection of the prime ideals of A. The ring A is called a Hilbert ring if every prime ideal is an intersection of maximal ideals, in which case, clearly, $N(A) = J(A)$. The definitions of $J(A)$ and $N(A)$ for a noncommutative ring are somewhat more complicated.

For a finite group G, $K[X; G]$ is semisimple, provided that G has no element of order p when char $K = p > 0$; this is Maschke's theorem and is fundamental for the classification of the representations of G (cf. [**6**, p. 244]). The statement remains true if, instead of being finite, G is taken to be an arbitrary abelian group (cf. [**4**, p. 73, Corollary 17.8]). To what extent does this latter result surface in the present book? The nearest theorem to it that I could find is Theorem 11.14, p. 140, which only yields the case that G is torsion-free. On the