BOOK REVIEWS

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Quadratic differentials, by Kurt Strebel, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge · Band 5, Springer-Verlag, Berlin, 1984, xii + 184 pp., $38.00. ISBN 3-5401-3035-7

Quadratic differentials have played an important role in complex analysis, particularly in the study of extremal problems. We start with the following elementary problem posed and solved by Grötzsch in the 1920s.

Suppose one is given two rectangles R and R' with sides parallel to the coordinate axes with lengths a, b and a', b', respectively. We wish to find an orientation-preserving diffeomorphism of R to R' mapping horizontal sides to horizontal sides and vertical sides to vertical sides as close to conformal as possible. This means minimizing

$$K(f)=\frac{1+\|f_{\bar z}/f_z\|_\infty}{1-\|f_{\bar z}/f_z\|_\infty}$$

over all possible f, where

$$f_{\bar z}=\tfrac{1}{2}(f_x+if_y) \quad \text{and} \quad f_z=\tfrac{1}{2}(f_x-if_y).$$

A map f such that $K(f)\leqslant K_0$ is called K_0-*quasiconformal.*

If the restriction on the sides is removed, the Riemann mapping theorem says there is a conformal map; $f_{\bar z}=0$ so $K=1$.

For the problem at hand, the affine map $f_0(z)=u_0(z)+iv_0(z)=(a'/a)x + i(b'/b)y$ satisfies

$$K(f_0)=\max(b'a/a'b,\, a'b/b'a).$$

Grötzsch proved that if f is any other map, then $K(f)>K(f_0)$. So the affine map is the unique extremal.

In the late 1930s Teichmüller found a far-reaching generalization of Grötzsch's result to compact Riemann surfaces. Suppose $f_0\colon R\to S$ is a homeomorphism between compact Riemann surfaces. We wish to minimize $K(f)$ over all f homotopic to f_0. Here the computations are in local analytic coordinates, and the number $|f_{\bar z}/f_z|$ does not depend on the choice of local coordinates at p or $f(p)$. Teichmüller found that there was a unique extremal map which locally is affine, with the affine coordinates provided by holomorphic quadratic differentials on R and S.

To explain this result more precisely, let us define what is meant by a quadratic differential. If Ω is a domain in the complex plane a holomorphic (resp. meromorphic) quadratic differential is a form $\phi(z)\,dz^2$ where $\phi(z)$ is a holomorphic (resp. meromorphic) function. If R is a Riemann surface, then a quadratic differential assigns to each analytic local coordinate a holomorphic (resp. meromorphic) function such that $\phi(z)\,dz^2$ is invariant under change of coordinates; that is, if w is another local coordinate with associated function $\psi(w)$ then

$$\psi(w)(dw/dz)^2 = \phi(z) \quad \text{in the overlap.}$$

A quadratic differential has a certain number of isolated zeroes and poles. What is particularly important from the point of view of Teichmüller's theorem are the vertical (resp. horizontal) trajectories of the quadratic differentials. These are curves C such that $\phi(z)\,dz^2 < 0$ (resp. > 0) along C. More precisely, for any parametrization $z = \gamma(t)$ of C,

$$\phi(\gamma(t))\gamma'(t)^2 < 0 \quad (\text{resp.} > 0).$$

The invariance property guarantees that this is well defined.

Teichmüller showed there was a unique map f in the homotopy class minimizing $K(f)$, which has the following form. There are holomorphic quadratic differentials ϕ on R and ψ on S, with $K > 1$, such that f maps zeroes of ϕ to zeroes of ψ of the same order, and maps horizontal and vertical trajectories to horizontal and vertical trajectories, stretching the horizontals by a factor $K^{1/2}$ and contracting the verticals by $K^{1/2}$. In local coordinates away from the zeroes this takes the following form. There are local coordinates $z = x + iy$ on R and $w = u + iv$ on S such that ϕ is dz^2 and ψ is dw^2 in these coordinates, and f is given by $u = K^{1/2}x$ and $v = K^{-1/2}y$. ϕ and ψ are called the initial and terminal quadratic differentials for the map. Conversely, given ϕ on R and $K > 1$, there is a Teichmüller map defined on R with initial quadratic differential ϕ and factor $K^{1/2}$.

Rigorous proofs of Teichmüller's theorem were given by Ahlfors (1954) and Bers (1960). Teichmüller's theorem became one of the cornerstones of the modern theory of Riemann surfaces, Teichmüller spaces, and quasiconformal mappings.

In 1976 Thurston introduced the concept of a measured foliation on a C^∞ surface M primarily to study diffeomorphisms of M. The leaves of a measured foliation are topologically the same as the horizontal trajectories of a quadratic differential, and there is a measure on each transversal which is invariant if one flows along leaves. Thurston showed that any irreducible homotopy class of diffeomorphisms (no diffeomorphism in the class restricts to a map of a simpler surface by cutting along one or more curves) has a canonical representative f, a unique pair of transverse foliations F_1 and F_2, and $\lambda > 1$ such that f preserves the leaves of F_i, but expands the measure of one by λ and contracts the other by λ. In the language of Teichmüller maps this can be rephrased by saying that F_1 and F_2 are the horizontal and vertical trajectories of a quadratic differential ϕ on some Riemann surface R, $f\colon R \to R$ is the Teichmüller map with expansion factor $K = \lambda^2$ and both initial and terminal quadratic differential ϕ.

In 1978 Bers gave a new proof of Thurston's classification of diffeomorphisms by posing and solving an extremal problem for self-homeomorphisms on Riemann surfaces.

Thurston's work on measured foliations and the companion notion of a geodesic lamination has been of fundamental importance in recent advances in surface topology and the geometry of 3-manifolds. These are extremely active areas of research. Also, quite recently, quadratic differentials and their trajectories have been directly applied to the ergodic theory of interval exchange transformations and flows on billiard tables whose angles are rational multiples of π. The theory of quadratic differentials with closed trajectories has been used to study the homology of the mapping class group and the moduli space of curves.

Most books on Riemann surfaces discuss quadratic differentials from the point of view of the Riemann-Roch theorem, that is, their dimension as a vector space and the number of zeroes. Up until now there has been no book with a detailed theory of their trajectory structures. In light of some of the developments mentioned above, the time is opportune for such a book, which brings us to the book under review. The author restricts himself mainly to fixing a Riemann surface and studying quadratic differentials on the surface. Thus there is no discussion of Teichmüller maps, nor is there any mention of the author's important work with Reich on quasiconformal mappings and quadratic differentials on the unit disc. This is perhaps unfortunate because one of the biggest motivations for studying quadratic differentials is absent. However, an excellent treatment of Teichmüller's theorem can be found in Abikoff's book. These limitations aside, this is a well-written book that will be of value to experts and anyone wishing to learn more about this important subject. The exposition is precise and detailed, yet not difficult to follow.

The book begins with a nice introductory chapter on some Riemann surface theory. Then the basic facts about quadratic differentials are developed both holomorphic and meromorphic as well as the local structure of trajectories near regular points, zeroes and poles, and then theorems on the global structure. A major part of the book is spent studying the important class of quadratic differentials whose horizontal trajectories are closed and the methods for approximating any quadratic differential by one with closed trajectories. This subject closely parallels Thurston's description of how a measured foliation is the limit of longer and longer simple closed curves. It can profitably be studied in conjunction with the theory of measured foliations found in [**5**].

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