BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 2, April 1985

# GEL'FAND'S PROBLEM ON UNITARY REPRESENTATIONS ASSOCIATED WITH DISCRETE SUBGROUPS OF $\mathrm{PSL}_2(\mathbf{R})$

BY TOSHIKAZU SUNADA[1]

In 1978 M.-F. Vignéras [**10**] gave a negative answer to the question posed by I. M. Gel'fand [**2**], who asked if the induced representation of $\mathrm{PSL}_2(\mathbf{R})$ on $L^2(\Gamma\backslash\mathrm{PSL}_2(\mathbf{R}))$ determines a discrete subgroup $\Gamma$ up to conjugation. She constructed explicitly two nonconjugate discrete groups arising from indefinite quaternion algebras defined over number fields giving rise to isomorphic induced representations. Such examples are necessarily quite sporadic since there are only finitely many conjugacy classes of arithmetic groups with a fixed signature; see K. Takeuchi [**9**]. The purpose of this note is to give, in a rather simple way, a large family of (nonarithmetic) discrete groups that are not determined by their induced representations. The key idea is to reduce the problem to the case of finite groups where the situations are simple and well understood. We should point out that a similar idea can be applied to constructions of various isospectral Riemannian manifolds [**8**].

Our construction is based on the following proposition, which follows from standard facts about induced representations.

PROPOSITION. *Let $G$ be a locally compact topological group, and let $\Gamma \subset \Gamma_1,\ \Gamma_2 \subset \Gamma_0$ be discrete subgroups, with $\Gamma$ normal and of finite index in $\Gamma_0$. Then if the subgroups $\mathcal{H}_i = \Gamma_i/\Gamma$, $i = 1, 2$, of $\mathcal{G} = \Gamma_0/\Gamma$ meet each conjugacy class of $\mathcal{G}$ in the same number of elements, the representations of $G$ on the spaces $L^2(\Gamma\backslash G)$, $i = 1, 2$, are unitarily equivalent.*

To construct nonconjugate $\Gamma_1$ and $\Gamma_2$ in $\mathrm{PSL}_2(\mathbf{R})$, we first choose an appropriate triple $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2)$ with the same induced representation $\mathrm{Ind}_{\mathcal{H}_1}^{\mathcal{G}}(1) \equiv \mathrm{Ind}_{\mathcal{H}_2}^{\mathcal{G}}(1)$. For instance, we let $\mathcal{G}$ be the semidirect product $(\mathbf{Z}/8\mathbf{Z})^{\times} \cdot (\mathbf{Z}/8\mathbf{Z})$ and set $\mathcal{H}_1 = \{(1,0),(3,0),(5,0),(7,0)\}$, $\mathcal{H}_2 = \{(1,0),(3,4),(5,4),(7,0)\}$. It is easy to check that $\mathcal{H}_1$ and $\mathcal{H}_2$ are not conjugate in $\mathcal{G}$, and each conjugacy class of $\mathcal{G}$ meets $\mathcal{H}_1$ and $\mathcal{H}_2$ in the same number of elements.

We then take a torsion-free discrete subgroup $\Gamma_0 \subset \mathrm{PSL}_2(\mathbf{R})$ satisfying the following conditions:

(i) the genus of the Riemann surface $\Gamma_0\backslash\mathrm{PSL}_2(\mathbf{R})/\mathrm{SO}(2)$ is greater than two;

(ii) $\Gamma_0$ is maximal in $\mathrm{PGL}_2(\mathbf{R})$; and

(iii) $\Gamma_0$ is nonarithmetic.

Since the fundamental group of a Riemann surface of genus $k$ has a free group on $k$ generators as quotient, we may always, if $k$ is large enough, find a sur-

Received by the editors May 8, 1984 and, in revised form, September 17, 1984.
1980 *Mathematics Subject Classification*. Primary 22E40, 22D10; Secondary 22E46, 20H10, 22D30.
[1]Supported by the Ishida Foundation.

jective homomorphism $\varphi\colon \Gamma_0 \to \mathcal{G}$. Since the example of $\mathcal{G}$ given in the previous paragraph has three generators, for it we may take $k = 3$. If we put $\Gamma_i = \varphi^{-1}(\mathcal{H}_i)$, then, in view of the Proposition, $\Gamma_1$ and $\Gamma_2$ give rise to isomorphic representations. But $\Gamma_1$ and $\Gamma_2$ are not conjugate in $PGL_2(\mathbf{R})$. In fact, if $g\Gamma_1 g^{-1} = \Gamma_2$ for some $g \in PGL_2(\mathbf{R})$, then $g\Gamma_0 g^{-1}$ is commensurable with $\Gamma_0$. A result announced by G. A. Margoulis [**6**] implies that the commensurability group $C(\Gamma_0) = \{h \in PGL_2(\mathbf{R}); h\Gamma_0 h^{-1}$ is commensurable with $\Gamma_0\}$ is discrete provided that $\Gamma_0$ is nonarithmetic. Since $C(\Gamma_0) \supset \Gamma_0$, from the maximality of $\Gamma_0$, it follows that $g \in \Gamma_0$; hence, $\varphi(g)\mathcal{H}_1\varphi(g)^{-1} = \mathcal{H}_2$. This is a contradiction.

Existence of "many" $\Gamma_0$ satisfying (i)–(iii) is guaranteed by results of L. Greenberg [**4**], A. M. Macbeath and D. Singerman [**5**], and K. Takeuchi [**9**]. In fact, by [**4**], generic $\Gamma_0$ are maximal in $PSL_2\mathbf{R})$. If such a $\Gamma_0$ is not maximal in $PGL_2(\mathbf{R})$, then the normalizer $N(\Gamma_0)$ of $\Gamma_0$ in $PGL_2(\mathbf{R})$ is strictly bigger than $\Gamma_0$, so that the isometry group $N(\Gamma_0)/\Gamma_0$ of the surface $\Gamma_0\backslash PSL_2(\mathbf{R})/SO(2)$ is not trivial. On the other hand, by [**5**], the isometry group is trivial for generic $\Gamma_0$. Combining these facts with finiteness of arithmetic groups [**9**], we get the genericity of $\Gamma_0$ satisfying (i)–(iii).

REMARK. (a) For the above groups $\Gamma_i$ the genus of the surface $\Gamma_i\backslash PSL_2(\mathbf{R})/SO(2)$ equals $8k - 7$ $(k \geq 3)$. The examples given by Vignéras have much bigger genus.

(b) Examples of $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2)$ satisfying the condition in the Proposition have been used by number-theorists to construct nonisomorphic number fields with the same Dedekind zeta function (see, for instance, [**7**]). It is also known that many examples of the triple $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2)$ arise from simple algebraic groups: If $\mathcal{G}$ is a reductive algebraic group and $\mathcal{H}_1, \mathcal{H}_2$ are nonconjugate but associate parabolic subgroups, then $\mathrm{Ind}_{\mathcal{H}_i}^{\mathcal{G}}(1)$, $i = 1, 2$, are isomorphic.

We thank the referee for his valuable suggestions.

## REFERENCES


1. F. Bruhat, *Sur les representations induits des groupes de Lie,* Bull. Soc. Math. France **84** (1956), 97–205.
2. I. M. Gel'fand, *Automorphic functions and the theory of representations,* Proc. Internat. Congress Math. (Stockholm, 1962), pp. 74–85.
3. I. M. Gel'fand, M. I. Graev and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions,* Saunders, Philadephia, 1969.
4. L. Greenberg, *Maximal Fuchsian groups,* Bull. Amer. Math. Soc. **69** (1963), 569–573.
5. A. M. Macbeath and D. Singerman, *Spaces of subgroups and Teichmüller spaces,* Proc. London Math. Soc. **31** (1975), 211–256.
6. G. A. Margoulis, *Discrete groups of isometries of manifolds of nonpositive curvature,* Proc. Internat. Congress Math. (Vancouver, 1974), Vol. 2, pp. 24–34.
7. R. Perlis, *On the equation* $\zeta_k(s) = \zeta_{K'}(s)$, J. Number Theory **9** (1977), 342–260.
8. T. Sunada, *Riemannian coverings and isospectral manifolds,* Ann. of Math. (to appear).
9. K. Takeuchi, *Arithmetic Fuchsian groups with signature* $(1, e)$, J. Math. Soc. Japan **35** (1983), 381–407.
10. M.-F. Vignéras, *Examples de sous-groupes discrets no-conjugués de* $PSL(2, \mathbf{R})$ *qui ont même la fonction zêta de Selberg,* C. R. Acad. Sci. Paris Sér A-B **287** (1978), 47–49.
11. ——, *Variétés riemanniennes isospectrales et nonisometriques,* Ann. of Math. (2) **112** (1980), 21–32.


DEPARTMENT OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA 464, JAPAN