this case asserts that if A is a complete local ring with maximal ideal m and if the characteristic of A/m coincides with that of A, then there is a subfield C of A such that C forms a complete set of representatives for A/m. Thus, A is a homomorphic image of a formal power series ring over the field C.

Cohen's structure theorem in these forms was a remarkable development in the theory of local rings, and some of the results derived from it are given in §4. Namely, §4 contains applications of the structure theorem to the theories of Japanese rings and Nagata rings; a Japanese ring is a neotherian integral domain A such that for any finite algebraic extension L of its field of fractions, the integral closure of A in L is a finite A-module. A Nagata ring is a noetherian ring A such that, for any prime ideal P of A, A/P is a Japanese ring, namely a pseudogeometric ring in the sense of Nagata [**8**]. At the end of the chapter there is an appendix in which a special type of extension of a local ring—roughly speaking, residue field extension—is discussed. If A is a local ring with maximal ideal M and if B is an extension discussed here, then MB is the maximal ideal of B and B is flat over A.

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Classical potential theory and its probabilistic counterpart, by J. L. Doob, Grundlehren der mathematischen Wissenschaften, vol. 262. Springer-Verlag, New York 1984, xxiii + 846 pp., $58.00. ISBN 0-3879-0881-1

It had been known for more than ten years that Doob was writing a book on this subject. Now that it has appeared, it entirely fulfills our expectations: it is a great work. Great by its dimensions, written with extreme love and care, concentrating the knowledge of a generation which was supreme in the history of potential theory, it also represents the achievement of Doob's own epoch-making research on the relations between classical potential theory and the theory of Brownian motion.

All this may be obvious for the specialist, but I suppose that for the layman who takes in hand this book for the first time, the question will be, How is it possible to write such a book (840 pages!) on a subject which now looks like a tiny place in the wide field of analysis, lost somewhere between elliptic partial differential equations and complex variables? Therefore, it may be reasonable, before giving details about the book, to say something about potential theory itself.[1]

The most typical problem in classical potential theory is that of the electrostatic condenser. Imagine a hollow conductor C as the boundary of a bounded domain Ω in $\mathbf{R}^3$, inside which is placed another conductor F (calling F a conductor will not imply, in the mathematical representation, that F is connected in the topological sense: it is simply a closed set contained in Ω). Then C is grounded, and F is wired to the positive pole of an electrostatic generator. Experience shows that negative charges will appear on the inner surface of C and positive charges on the outer surface of F, balancing each other in such a way that an *equilibrium potential* V is generated within Ω (assuming on C the constant value 0 (by convention), on F some constant positive value (for mathematicians, the value 1)) and harmonic in $\Omega \setminus F$. The total positive charge on F is the condenser's *capacity*.

Turning this experimental evidence into rigorous mathematics has been a challenge for more than a century, starting with Gauss (1840) and attracting the interest of such people as Riemann, Weierstrass, Dirichlet, Schwarz, Poincaré, Hilbert, and later, Lebesgue, La Vallée Poussin, F. and M. Riesz, Wiener, and many more. This problem has been a test for every new discovery in analysis: Hilbert space, integral equations, Lebesgue's integral with respect to arbitrary measures, distribution theory,... It has provided the motivation for an incredible amount of research in analysis (including such general theories as Choquet's capacities and integral representations in convex cones), and it isn't dead yet: significant work has been done very recently on double-layer potentials, a favourite tool of 19th century analysts.

The complete solution of the condenser problem was, after much preliminary work in the years around 1920, one of the achievements of the great period of classical potential theory, marked by the contributions of Frostman (1935), H. Cartan (active on this subject 1941–1946), M. Brelot[2] (from about 1938 to 1950, after which his work shifts to more general situations), R. S. Martin (1941; this paper went almost unnoticed until much later). You will find all this in Doob's book, which also contains some of Choquet's capacity theory (1951–1955) and, of course, Doob's own probabilistic interpretations (beginning 1954). After this period, in accordance with the spirit of the times, potential theory moved in the direction of generalization and axiomatization: Choquet's and Deny's search for kernels satisfying the basic "principles" of potential theory, and, in particular, Deny's "elementary kernels", which are the

[1] See the very interesting accounts of the history of potential theory by M. Brelot, Ann. Inst. Fourier (Grenoble) **4** (1952) and Enseign Math. (2) **18** (1972).

[2] It is a shame that Brelot's complete work has not yet been collected and his famous 1959 lectures have not been reprinted in English!

analytic version of the potential theory for Markov chains; Beurling's and Deny's "Dirichlet spaces", Brelot's "harmonic spaces", i.e. the axiomatization of potential theory by means of sheaves of harmonic functions, developed in different directions by Bauer and Constantinescu and Cornea. One should add to this the great probabilistic synthesis accomplished by Hunt (1957–1958) and the work of Dynkin's school, which is more oriented towards the probabilistic theory of Markov processes. Though Doob contributed personally to these developments, his title is explicitly *classical* potential theory, and from all that he has included only some of the probabilistic advances.[3]

On the other hand, the solution of the condenser problem and the general methods which are necessary for it occupy a central position in the book. Let us roughly describe them. The two conductors are treated in a somewhat asymmetric way. One first forgets about F. A unit positive charge at $x \in \Omega$ would generate in open space a potential $n(\cdot, x)$ (newtonian or Coulomb potential), but, due to the hollow conductor C, on the surface of which negative charges will appear, it generates a smaller potential $g(\cdot, x)$ (Green potential) in Ω, which still is superharmonic positive in Ω, but in some sense should "vanish at the boundary". The main advance is the description of this without any reference to boundary behaviour: $g(\cdot, x)$ is just $n(\cdot, x)$ minus its *greatest positive harmonic minorant* in Ω, which is shown to exist without any smoothness assumption on the domain. Of course, the difficulty has been shifted to another place: (1) Does this generalized Green function really vanish at the boundary in any reasonable sense? (2) How can one describe explicitly the greatest harmonic minorant which has been subtracted? Does it really correspond to a distribution of charges on the boundary?

The first problem is solved by the distinction between *regular* boundary points, at which the Green function vanishes in the ordinary sense, and *irregular* points. These are characterized as the points where the (closed) complement of Ω is "thin", and they are shown to form a very scarce *polar set*. The exact description of these exceptional sets has been one of the major steps in the development of classical potential theory: after preliminary descriptions they could be characterized as sets of inner capacity 0. Then Cartan proved they had *outer* capacity 0, a much stronger result, which was finally understood when Choquet proved that inner and outer capacity are the same for nice (analytic and, in particular, borelian) sets and for much more general "capacities".

The second problem can be solved in two ways. One can show that the greatest harmonic minorant of $n(\cdot, x)$ is the newtonian potential of a probability measure $h(x, dy)$ on C, the *swept measure* of ε_x, or *harmonic measure* at x. It turns out that it is carried by the regular points of C and is the essential tool for the solution of the Dirichlet problem in Ω by the PWB (Perron-Wiener-Brelot) method. then $-h(x, dy)$ describes the negative charges which appeared "by influence" on the hollow conductor. On the other hand, one may forget about the open space and look for an integral representation of

[3] It may be worthwhile to emphasize that potential theory is alive and well today, though it isn't as popular as in the sixties.

all positive harmonic functions in Ω by charges distributed over an *ideal boundary*, which may be quite different from the euclidean one. This leads, in particular, to the Martin compactification and Martin boundary of Ω.

Two remarks are in order here. The first one concerns parabolic (heat equation) potential theory, to which Doob devotes a fair amount of his book. Most analysts before Doob had lived with the simple idea that elliptic equations are naturally associated with problems of Dirichlet type; parabolic and hyperbolic equations, with Cauchy problems. Since his paper of 1955 on the heat equation, Doob has insisted on the fact that from the probabilistic point of view, there is little difference between Laplace's equation and the heat equation, and therefore the heat equation may be treated in parallel with classical potential theory. The main difference is that the usual exceptional sets are now the so-called *semipolar* sets, which are not so scarce and cannot be entirely ignored.

The second remark is the striking interpretation of harmonic measure using brownian motion: if you place a brownian particle at x, then the harmonic measure $h(x, dy)$ can be described as the distribution of the (random) place where the particle first hits the boundary C. This, of course, requires a lot of work for rigorous justification, but it gives an extremely intuitive content to the abstract analysis of harmonic minorants, etc. Probabilistic potential theory as Doob sees it, however, is much more than an *interpretation* of classical potential theory. (The title says *counterpart*, which is quite different; we return to this below).

We have not yet placed the second conductor F inside C: we would like to prove the existence of an equilibrium potential V which vanishes at the boundary of Ω (hence, it is reasonable to expect for it a representation as a Green potential $\int g(\cdot, y)\,\mu(dy)$), is harmonic outside F (hence, μ is expected to be carried by F), and takes the value 1 on F. By maximum principle considerations, any positive superharmonic function which dominates 1 on F should dominate V everywhere in Ω. This led Brelot to study (forgetting again about potentials, vanishing at the boundary, etc.) the *reduction of* 1 *on* F, i.e., the infimum of the class of all positive superharmonic functions in Ω larger than 1 on F. It turns out that this function is indeed harmonic in $\Omega \setminus F$ and is equal to the equilibrium potential V we are trying to construct, except on the polar set consisting of those $x \in F$ where F is "thin". Again brownian motion provides a striking interpretation: $V(x)$ is just the probability that, for brownian motion starting from x, F will be hit before C.

However, the key result of probabilistic potential theory is the fact that superharmonic functions, which for the analyst are quite irregular lower semicontinuous functions, are seen by the probabilist as *continuous* functions along brownian paths. This was discovered by Doob in 1954, extended by him to the parabolic case in 1955 (continuity being replaced by right continuity with left limits), and opened the main way of communication between continuous time martingale and supermartingale theory and potential theory, through which they influenced each other.

Slightly more than half of Doob's book is devoted to pure analysis: classical potential theory (a complete treatise in itself) and parabolic potential theory (the only existing account of this subject). The other half is the "counterpart": a self-contained exposition of the fundamentals of stochastic processes, martingales and supermartingales (including the decomposition theory), brownian motion, and related processes. A limited amount of stochastic integration and Markov processes is also included, sufficient for the purpose of the book. The stress in probabilistic potential theory is laid on boundary behaviour: additive functionals are deliberately left aside. It seems that somehow the overlap with books devoted to Hunt's general results (like Chung's *Lectures from Markov processes to Brownian motion*, to mention only a recent one) has been minimized. On the other hand, the development is remarkably free of heavy technicalities, and the counterpoint with analytic potential theory is fascinating. It oscillates between full symmetry and tiny analogies which remind us (likie the finger remnants of a whale) that superharmonic functions and supermartingales have a common origin in superaveraging properties.

The book is not only great as a whole, it also seems perfect in every detail. The index is unusually complete and precise, the printing wide and beautiful. A malevolent search for misprints over many pages caught only two, so insignificant that I would be ashamed to quote them.[4] The style is, in the reviewer's opinion, very attractive, rather explanatory than dogmatic.

P. A. MEYER

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[4] No first edition of any book, however, can be entirely free of mathematical errors. The author kindly communicated to me his own errata list, which *does* contain a few misprints (let me add to them the fact that the definition of "nth entry time" on p. 420 is slightly incorrect, though the following lines explain it well) and also points out a more serious error related to the notion of *accessible* time (the reviewer feels quite sympathetic, since he once fell into the same trap). It begins on p. 430 and the example on p. 431: it is not true that accessible graphs are unions of predictable graphs; they are just contained in such a union. This implies the modification of the lower half of p. 487 and of most of pp. 498–499. So the reader of the purely probabilistic part of the book should be a little careful until the errata list is published (the potential theoretic part, either analytic or probabilistic, is entirely unaffected by this mistake). The reviewer would like to mention also that the phrase "quasi-left continuous filtration" is used by most authors instead of Doob's "predictable filtration" and should probably be added to the index.