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Groups of divisibilty, by Jiří Močkoř, Mathematics and its Applications, D. Reidel Publishing Company, Dordrecht, 1983, 184 pp., $39.50. ISBN 9-0277-1539-4

Associated with any integral domain D there is a partially ordered group $G(D)$, called the group of divisibility of D. When D is a valuation domain, G is merely the value group; and in this case, ideal-theoretic properties of D are easily derived from corresponding properties of G, and conversely. Even in the general case, though, it has frequently proved useful to phrase a ring-theoretic problem in terms of the ordered group G, first solve the problem there, and then pull back the solution if possible to D. Lorenzen originally applied this technique to solve a problem of Krull, and Nakayama used it to produce counterexamples to another problem of Krull and to a related problem of Clifford. Thus, the basic idea is to use partially ordered groups to produce examples of domains, the advantage deriving from the fact that partially ordered groups abound, whereas domains are not so easy to come by.

The definition of $G(D)$ for a domain D with quotient field K runs as follows. A principal fractional ideal of D is a set of the form aD, $0 \neq a \in K$, and is denoted by (a). These ideals form a group $G(D)$ under the multiplication $(a)(b) = (ab)$. Sometimes the ideal (0) is thrown in and denoted by ∞. An equivalent way of thinking of $G(D)$ is to identify elements of K which differ by unit multiplies from D, i.e., $G(D) = K^*/U(D)$, where K^* denotes the multiplicative group of nonzero elements of K, and $U(D)$ denotes the multiplicative group of units in D. In spite of its multiplicative origin, G is usually written additively, probably because of a desire to picture it graphically; but in most treatments, and the present book is no exception, there is a certain amount of vacillation between multiplicative and additive notation. The partial ordering on G is defined by $(a) \geqslant (b)$ if $(a) \subset (b)$, i.e., by reverse inclusion (think of $(4) \geqslant (2)$ when D is the ring of integers).

For a simple example take D to be a UFD. Then $G(D)$ is the direct sum of I copies of Z, the additive group of integers, where the index set I has the same cardinality as the set of principal prime ideals of D and the ordering is coordinatewise. This is merely a reflection of the fact that for any nonzero element $a \in K$, the fractional ideal (a) may be uniquely written in the form $(a) = (p_1)^{i_1} \cdots (p_n)^{i_n}$, where $i_j \in Z$, and $p_1, \ldots, p_n$ are distinct irreducible elements of D. In particular, the G arising from a UFD is a lattice ordered group, so it is somewhat surprising to discover

THEOREM. *A* (*p.o. abelian*) *group G is lattice ordered* (*if and*) *only if it is the group of divisibility of a Bezout domain.*

A bezout domain is one for which finitely generated ideals are principal (thus, the noetherian Bezout domains are just the PIDs). The author calls this

Jaffard-Ohm's theorem. Its main assertion, due to Jaffard, says that a lattice ordered group is a group of divisibility, while the observation that Jaffard's construction actually gives a Bezout domain is the reviewer's. (Kaplansky also apparently proved Jaffard's part in his thesis but never published it.) This theorem is the subject of Chapter 8, where the author gives two proofs of it: one is the original proof of Jaffard; the other is a more conceptual proof due to the reviewer. He also includes in Chapter 12 a number of well-chosen applications of the theorem.

It would be useful to have a result of this type that applies to a more general class of partially ordered groups than the lattice ordered groups. A step in this direction is the theorem of [**1**], which asserts that a lex extension of a group of divisibility by a totally ordered group is again a group of divisibility. (Lex extensions are discussed in Chapter 9 of the present book, but [**1**] is too recent to be included.)

The first seven chapters of this book are devoted to developing properties of groups of divisibility and connecting these to corresponding properties of domains. Much of this is done in the context of the so-called "d-groups", a concept originally due to T. Nakano and developed further by the author. This is a notion halfway between D and $G(D)$ and is designed to carry over to $G(D)$ some of the additive information from D. Thus, instead of merely regarding $G(D)$ as a p.o. group, one introduces a map $\oplus$ from $G \times G$ to the set of all subsets of G which satisfies certain axioms. An example of such a $\oplus$ is obtained by using the addition $+$ of D to define $(a) \oplus_D (b) = \{(c) \mid$ there exist $u_1, u_2 \in U(D)$ with $c = u_1 a + u_2 b\}$.

Some additional contents are a chapter on the group of divisibility of a Krull domain and its relationship to the class group, a chapter on topological groups of divisibility, and even a chapter containing a sheaf-theoretic treatment of approximation theorems.

To sum up, this book is intended to be a survey of recent results in this area (perhaps the last twenty years), directed toward specialists, and I believe the author succeeds well in this aim. The material is developed with sufficient attention to detail that it can easily be read by the general mathematical public, but its chief appeal will be to specialists.

REFERENCES

1. D. F. Anderson and J. Ohm, *Valuations and semi-valuations of graded domains*, Math. Ann. **256** (1981), 145–156.

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