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Differential forms orthogonal to holomorphic functions or forms, and their properties, by L. A. Aizenberg and Sh. A. Dautov, Translations of Mathematical Monographs, Vol. 56, American Mathematical Society, Providence, Rhode Island, 1983, ix + 165 pp., $36.00. ISBN 0-8218-4508-X

A considerable portion of complex analysis in several variables is devoted to developing n-dimensional analogs of classical one-variable theorems, and much of its fascination derives from the fact that these analogs often take forms which are subtle and surprising at first glance and seem natural only with hindsight. As a simple example, consider the fact that the zero set $N(f) = \{z: f(z) = 0\}$ of a holomorphic function of one variable consists of isolated points. This result, as stated, is utterly false in higher dimensions, for the zeros of holomorphic function of more than one variable are *never* isolated. But one obtains a correct theorem in any number of dimensions by rephrasing the one-variable result suitably: namely, $N(f)$ is an analytic subvariety of complex codimension one. Another example comes from the theorems of Mittag-Leffler and Weierstrass on finding meromorphic or holomorphic functions with prescribed poles or zeros; to obtain their n-dimensional analogs, the so-called Cousin problems, one should reformulate them in terms of sheaf cohomology.

The book of Aizenberg and Dautov takes as its starting point the following characterization of boundary values of holomorphic functions of one variable, which is well known to the experts but perhaps not to the general public. Let D be a bounded open set in $\mathbf{C}$ with smooth boundary $\partial D = \overline{D} \setminus D$; let $A(D)$ be the set of continuous functions on $\overline{D}$ which are holomorphic on D, and $A(\partial D) = \{f|\partial D: f \in A(D)\}$.

THEOREM 1. *For a continuous function f on ∂D to be in $A(\partial D)$ it is necessary and sufficient that*

$$(1) \qquad \int_{\partial D} f(z)g(z)\,dz = 0 \quad \text{for all } g \in A(\partial D).$$

The necessity is an immediate consequence of Cauchy's theorem. To prove the sufficiency, one defines a holomorphic function φ on $\mathbf{C} \setminus \partial D$ by plugging f into the Cauchy integral formula:

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w - z}\,dw.$$

It is not too hard to prove that the difference of the limits of $\varphi(z)$ as z approaches a point $z_0 \in \partial D$ along the normal to ∂D from the inside and from the outside equals $f(z_0)$. On the other hand, it follows from (1) that $\varphi(z) = 0$ for $z \notin \overline{D}$. One concludes that the function F on $\overline{D}$, defined by $F = \varphi$ on D and $F = f$ on ∂D, is continuous, so $f \in A(\partial D)$.

How can this result be generalized to n dimensions? We begin with a bounded open set D in $\mathbf{C}^n$ with smooth boundary ∂D. (The definition of "smooth" depends to some extent on how much trouble one wishes to go to in stating and proving one's theorems. Depending on context, it should mean at least C^1 and perhaps as much as C^∞.) To generalize (1) we should rephrase it in terms of differential forms, the natural objects to integrate over submanifolds, by incorporating the dz into f or g; "holomorphic" should then mean "$\bar\partial$-closed". We are thus led to the following general problem: Characterize those forms φ of bidegree (p, q) [i.e., degree p in the dz's and degree q in the $d\bar z$'s] on ∂D such that

$$(2) \qquad \int_{\partial D} \varphi \wedge \psi = 0 \quad \text{for all forms } \psi \text{ of bidegree } (n-p, n-q-1) \text{ with } \bar\partial\psi = 0.$$

This question was considered some twenty years ago by Kohn and Rossi [**1**], who proved an analog of Theorem 1: if $q < n - 1$ and certain pseudoconvexity conditions on D are satisfied, a (p, q)-form φ satisfies (2) if and only if it has a $\bar\partial$-closed extension to D. In the case $p = q = 0$ (when φ is a function), Weinstock [**2**] improved the result by weakening the smoothness conditions and removing the pseudoconvexity hypothesis.

Aizenberg and Dautov are concerned with the remaining case, $q = n - 1$. Here, the $\bar\partial$-closed forms of bidegree $(p, 0)$ entering in (2) are better known as holomorphic p-forms, and the condition that $\int_{\partial D} \varphi \wedge \psi = 0$ is what is meant in the title of their book by "orthogonal" (a slightly disconcerting usage for those of us accustomed to working with Hermitian inner products). They prove that the analog of Theorem 1—namely, a $(p, n-1)$-form φ satisfies (2) if and only if it has a $\bar\partial$-closed extension to D—holds when D is a strongly pseudoconvex domain, or, more generally, when D is obtained from such a domain by removing finitely many strongly pseudoconvex subdomains; they also give a similar but weaker result for more general D. Once this is established, the following line of thought suggests itself. In Theorem 1 the functions f and g enter symmetrically: in dimension one the space of holomorphic functions is self-orthogonal in the sense of (1). Might it therefore happen that some properties of holomorphic functions in one variable generalize most naturally not to holomorphic functions or forms in n variables but to their annihilators in the sense of (2), that is, the $\bar\partial$-closed $(p, n-1)$-forms? The answer is *yes*, and Aizenberg and Dautov present an interesting selection of results to prove the point, including analogs of Runge's and Morera's theorems and a representation of distributions on $\mathbf{R}^{2n-1}$ as "boundary values" of $(n, n-1)$-forms on half-spaces of $\mathbf{C}^n$.

This book was originally published in Russian in 1975 as a report on the research of the authors and some other Soviet mathematicians from the early 1970s. When the English translation was proposed in 1981, the authors provided some additional chapters which discuss more recent results and which now make up about forty percent of the book. Written clearly but in the no-nonsense style of a research monograph, this book provides a rewarding look at some of the recent work of the Soviet school of complex analysis in several variables for those with some previous experience in the subject.

REFERENCES

1. J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math. (2) **81** (1965), 451–472.

2. B. M. Weinstock, *Continuous boundary values of analytic functions of several complex variables*, Proc. Amer. Math. Soc. **21** (1969), 463–466.

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Univalent functions, by Peter L. Duren, Grundlehren der mathematischen Wissenschaften 259, Springer-Verlag, New York, 1983, $46.00, xiv + 382 pp. ISBN 0-3879-0795-5

Prefatory Note (added August, 1984). After the review appearing below was submitted I learned that Bieberbach's conjecture had been proved by L. de Branges. His proof is short and miraculous. It combines the theories of Loewner and Milin with a new ingredient from a totally unexpected source: a theorem of Askey and Gasper (Amer. J. Math. **98** (1976), 709–737, Theorem 3) which asserts that

$$\sum_{j=0}^{k} P_j^{(\alpha,0)}(x) > 0$$

for $-1 < x \leqq 1$ and $\alpha > -2$, where $P_j^{(\alpha,\beta)}$ denote the Jacobi polynomials.

Thus, some of the discussion of Bieberbach's conjecture below is obsolete, except insofar as it can serve to showcase the remarkableness of de Branges' achievement. Although its most famous problem has now been solved, the subject of univalent functions remains interesting, both for its own sake and for its connections with other branches of analysis, and Duren's book is an outstanding contribution to it.

In the language of classical complex function theory, "univalent" means one-to-one. Thus, the univalent functions of Duren's book are analytic functions which are one-one in some connected open subset of the complex plane, most often the unit disk $\mathbf{D} = \{z \in \mathbf{C}: |z| < 1\}$. Such functions effect a conformal mapping onto another domain $\Omega \subset \mathbf{C}$.

Much research in the subject, and most of this book, is devoted to the class S of univalent analytic functions f in $\mathbf{D}$ which satisfy the normalizations