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A SYMPLECTIC FIXED POINT THEOREM FOR COMPLEX PROJECTIVE SPACES

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1. Arnold's conjecture. An automorphism ψ of a symplectic manifold (P, ω) is *homologous to the identity* if there is a smooth family ψ_t $(t \in [0,1])$ of automorphisms such that the time-dependent vector field ξ_t defined by $d\psi_t/dt = \xi_t \circ \psi_t$ is globally hamiltonian; i.e. if there is a smooth family H_t of real-valued functions on P such that $\xi_t \rfloor \omega = dH_t$. It was conjectured by Arnold [**1**], as an extension of the Poincaré-Birkhoff annulus theorem [**3**, **7**], that every automorphism of a compact symplectic manifold P, homologous to the identity, has at least as many fixed points as a function on P has critical points.

Arnold's conjecture was proven by Conley and Zehnder [**4**] for the torus $T^{2n} \approx \mathbf{R}^{2n}/\mathbf{Z}^{2n}$ with its usual symplectic structure. They show that every symplectic automorphism of T^{2n}, homologous to the identity, has at least $n+1$ fixed points, and at least 2^{2n} if all are nondegenerate. Their method was extended in [**8**] to prove a version of Arnold's conjecture for arbitrary P under the additional assumption that the hamiltonian vector field ξ_t is sufficiently C^0 small.

In this note we announce a proof of Arnold's conjecture for the complex projective space $\mathbf{C}P^n$ with its standard symplectic structure. We prove that a symplectic diffeomorphism of $\mathbf{C}P^n$, homologous to the identity, has at least $n+1$ *distinct* fixed points. (By the Lefschetz fixed point theorem, any continuous map from $\mathbf{C}P^n$ to itself, homotopic to the identity, has at least $n+1$ fixed points *counted with multiplicities.*) For $n = 1$ ($\mathbf{C}P^1 \approx S^2$) the result was already known [**1**], but with a proof which worked only in this two-dimensional case.

The proof for T^{2n} in [**4**] made use of a variational principle in which the fixed points of the map were identified with periodic solutions of a time-dependent hamiltonian system and then identified with critical points of a functional on the space of contractible loops on T^{2n}. The corresponding functional in the case of $\mathbf{C}P^n$ is multiple valued, and there are other difficulties connected with the curved geometry of $\mathbf{C}P^n$, so we need a new approach. Our trick is to consider the hamiltonian system on $\mathbf{C}P^n$ as the *reduction*, in the sense of [**6**], of a hamiltonian system on $\mathbf{C}^{n+1}$ and then adapt recently developed methods [**2**] for finding periodic orbits in $\mathbf{C}^{n+1}$. This method is similar to that of Conley and Zehnder in that a problem on a compact manifold is lifted to a problem on euclidean space invariant under a group of transformations.

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2. Lifting to $\mathbf{C}^{n+1}$. Consider $\mathbf{C}^{n+1}$ with its usual symplectic structure $\operatorname{Im}\sum dz_\imath \wedge d\bar{z}_\imath$. The hamiltonian $K(z) = \sum z_\imath \bar{z}_\imath$ generates the periodic flow $T_\mu(z_1, \ldots, z_{n+1}) = (e^{2\imath\mu}z_1, \ldots, e^{2\imath\mu}z_n)$ with period π, and hence an action of $S^1 = \mathbf{R}/\pi\mathbf{Z}$ (the Hopf fibration). The reduced manifold $K^{-1}(1)/S^1$ can be identified with $\mathbf{C}P^n$, and any S^1-invariant hamiltonian system on $\mathbf{C}^{n+1}$ induces a system on $\mathbf{C}P^n$, called the reduced system. Our idea is to use this procedure in the opposite direction.

Fixed points of $\psi\colon \mathbf{C}P^n \to \mathbf{C}P^n$ are the same as solution curves $\tilde{\sigma}\colon [0,1] \to \mathbf{C}P^n$ with $\tilde{\sigma}(0) = \tilde{\sigma}(1)$ for the time-dependent hamiltonian system which generates the family ψ_t connecting the identity to ψ. Let $\tilde{H}_t$ be the hamiltonian family for this system; since each $\tilde{H}_t$ contains an arbitrary constant, we may assume that $\tilde{H}_t(x) > 0$ for all t in $[0,1]$ and all x in $\mathbf{C}P^n$. Now let $H_t\colon \mathbf{C}^{n+1} \to \mathbf{R}$ be the unique function which is homogeneous of degree 2 and whose restriction to $K^{-1}(1) = S^{2n+1}$ is the pullback of $\tilde{H}_t$. Then H_t is S^1-invariant and defines a time-dependent hamiltonian system on $\mathbf{C}^{n+1}$ whose reduced system is H_t.

By the general theory of reduction, we know that S^{2n+1} is an invariant manifold for H_t, and the orbits of $\tilde{H}_t$ on $\mathbf{C}P^n$ are the images of orbits of H_t on S^{2n+1}. Furthermore, if $\tilde{\sigma}$ is the image of σ, then $\tilde{\sigma}(1) = \tilde{\sigma}(0)$ if and only if $\sigma(1) = T_\mu\sigma(0)$ for some μ in $\mathbf{R}/\pi\mathbf{Z}$. If we change the hamiltonian H_t to $H_t + \lambda K$ for some $\lambda \in \mathbf{R}$, then the "flow" of $H_t + \lambda K$ will still project to that of $\tilde{H}_t$, but now by choosing $\lambda \pmod{\pi} = \mu$ we can make $\sigma(1) = \sigma(0)$. In other words, to each closed solution curve $\tilde{\sigma}$ for $\tilde{H}_t$ and, hence, to each fixed point of ψ there corresponds a collection of pairs (σ, λ) where $\lambda \in \mathbf{R}$ and σ is a closed solution curve for $H_t + \lambda K$ on S^{2n+1}. The set of all pairs (σ, λ) corresponding to a given fixed point is diffeomorphic to $S^1 \times \mathbf{Z}$.

By Hamilton's principle the closed solution curves for $H_t + \lambda K$ on $\mathbf{C}^{n+1}$ are exactly the critical points of the functional

$$\begin{aligned} g(z) &= \int_0^1 -i(z'(t), z(t))\,dt + \int_0^1 H_t(z(t))\,dt + \lambda \int_0^1 |z(t)|^2\,dt \\ &= A(z) + H(z) + \lambda K(z). \end{aligned}$$

Since we are interested in critical points for all possible values of λ, we may consider λ as a Lagrange multiplier and look for critical points of $f(z) = A(z) + H(z)$ constrained to the infinite-dimensional sphere $K^{-1}(1)$.

We are thus faced with two problems. The first is to do the analysis which shows that $f(z)$ has many critical points on $K^{-1}(1)$, and the second is to show that all these critical points cannot belong to fewer than $n+1$ families of type $S^1 \times \mathbf{Z}$ coming from distinct fixed points of ψ.

3. Critical point analysis. The solution of the problems stated at the end of §2 forms the content of [**5**] and will only be summarized briefly here.

It turns out that the critical point theory developed in [**2**], based on the notion of *relative index*, is applicable to our problem, with some modifications made to permit working on the sphere $K^{-1}(1)$ within the space of loops of Sobolev class $H^{1/2}$ in $\mathbf{C}^{n+1}$. The values of the Lagrange multiplier λ are then found to be equal to the critical values of the functional f on $K^{-1}(1)$.

The minimax nature of the critical point theory makes it possible to estimate these values by comparison with the action functional A. A combinatorial argument then shows that these critical values cannot lie in less than $n+1$ cosets of $\mathbf{R}$ ($\bmod\, \pi\mathbf{Z}$) unless some critical values merge, in which case ψ would have uncountably many fixed points.

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