BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

C^*-ALGEBRAS AND DIFFERENTIAL TOPOLOGY

BY JEROME KAMINKER[1]

Let M be a smooth closed manifold. If D is an elliptic differential operator on M, then the differential structure on M is explicitly involved in the definition of the analytic index of D. It is a consequence of the Atiyah-Singer Index Theorem that this integer only depends on the homeomorphism type of the manifold M, since the topological formula for the index involves the rational Pontrjagin classes which are topological invariants.

By considering families of operators one may determine a more refined index for an elliptic operator which will lie in $K_0(M)$ [**1**]. This raises the possibility of torsion (i.e., finite order) invariants for operators. We exploit this to study the dependence of the algebra of 0th-order pseudodifferential operators on the underlying differential structure.

The BDF theory of C^*-algebra extensions [**2**] provides a formalism for studying such questions. Recall that the algebra of 0th-order pseudodifferential operators on a manifold $\mathcal{P}_0$ defines an extension of C^*-algebras $0 \to \mathcal{K} \to \mathcal{P}_0 \to C(SM) \to 0$, where SM is the tangent sphere bundle of M. We denote this by $\mathcal{P}_M \in \operatorname{Ext}(SM)$. There is a natural isomorphism $\Gamma\colon \operatorname{Ext}(SM) \to K_1(SM)$. Since SM is a Spin^c manifold, there is a topologically defined K-theory fundamental class $[SM] \in K_1(SM)$.

THEOREM 1. *The map* $\Gamma\colon \operatorname{Ext}(SM) \to K_1(SM)$ *satisfies* $\Gamma(\mathcal{P}_M) = [SM]$.

This follows from the index theorem for families of operators [**5**].

We now study the question of whether $\mathcal{P}_M$ depends on the smooth structure on M. Recall that the isotopy classes of smooth structures on M can be made into a finite abelian group $\mathcal{S}(M)$. We denote by M_α the manifold M with the differential structure $\alpha \in \mathcal{S}(M)$. The identity map $1\colon M_\alpha \to M$ induces a map $\overline{1}\colon SM_\alpha \to SM$. There is a unit, $u \in K^0(SM)$, such that $\overline{1}_*([SM_\alpha]) = u \cap [SM]$. Further, there is a unit $\theta(\alpha) \in K^0(M)$, depending only on the class of $\alpha \in \mathcal{S}(M)$, which is a lift of u in the sense that $\pi^*(\theta(\alpha)) = u$, where $\pi\colon SM \to M$ is the projection.

Thus, θ defines a map from $\mathcal{S}(M)$ to $K^0(M)$.

THEOREM 2 [**5**]. *The function* $\theta\colon \mathcal{S}(M) \to K^0(M)$ *is a homomorphism of* $\mathcal{S}(M)$ *into the multiplicative group of units* $1 \oplus \tilde{K}^0(M) \subseteq K^0(M)$.

The next step is to interpret θ homotopy theoretically. Here one must work separately on the 2-primary and odd-primary parts of $\mathcal{S}(M) = \mathcal{S}(M)_{(2)} \oplus \mathcal{S}(M)_{(\mathrm{odd})}$. The two analyses proceed in a parallel way, so we sketch only

Received by the editors March 29, 1984.
1980 *Mathematics Subject Classification*. Primary 58G15, 58G10; Secondary 46M20, 57R55.
[1]Research partially supported by NSF Grant MCS-81-02246.

the 2-primary case. (In [**5**] the odd-primary case was handled by a different method.)

Note first that $\mathcal{S}(M) \cong [M, \mathrm{Top}/O]$. A map $\alpha \colon M \to \mathrm{Top}/O$ can be interpreted as a vector bundle E along with a topological trivialization. Composing α with the natural map into G/O followed by the complexification of Sullivan's map $e \colon G/O \to BO^{\otimes}$ yields a unit comparing two orientations of E. This defines a homomorphism $e_{\mathbf{C}} \colon \mathcal{S}(M) \to K^0(M)$ mapping into the multiplicative group of units of $K^0(M)$.

THEOREM 3 [**6**]. *Let* $\alpha \in \mathcal{S}(M)$.
(i) *If* $\alpha \in \mathcal{S}(M)_{(\mathrm{odd})}$, *then* $\theta(\alpha) = e_{\mathbf{C}}(\alpha)^2$.
(ii) *If* $\alpha \in \mathcal{S}(M)_{(2)}$ *and, moreover,* M *is* 2-*connected, then* $\theta(\alpha) = e_{\mathbf{C}}(\alpha)^2$.

It follows from (i) and the odd-primary analysis of the fibration

$$\mathrm{Top}/O \xrightarrow{i} G/O \xrightarrow{j} G/\mathrm{Top} \tag{1}$$

due to Sullivan [**9**] that we have

THEOREM 4. *If* $\alpha \in \mathcal{S}(M)_{(\mathrm{odd})}$, *then* $\theta(\alpha) = 1$.

The 2-primary case is different. Here, we use the analysis of (1) localized at 2 due to Brumfiel, Madsen and Milgram [**3**]. We construct a finite complex X and a map $\mathfrak{a} \colon X \to \mathrm{Top}/O$ for which $(e_{\mathbf{C}} i \mathfrak{a})$ is not null-homotopic. By embedding X in a sphere and taking the double of a smooth regular neighborhood, one obtains a smooth manifold M. Using this manifold and the smooth structure determined by the map $\rho\mathfrak{a}$, where ρ is a retraction of M onto X, we obtain the following theorem.

THEOREM 5 [**6**]. *There is a smooth manifold* M *with a second differential structure* $\alpha \in \mathcal{S}(M)_{(2)}$, *for which* $\theta(\alpha) \neq 1$.

Thus P_M can, indeed, depend on the smooth structure.

COROLLARY 6. *The algebra of* 0*th-order pseudodifferential operators on* M *depends on the differential structure.*

Our construction yields an infinite family of such manifolds. However, one may also construct manifolds M and smooth structures in the 2-primary part of $\mathcal{S}(M)$ for which the invariant $\theta(\alpha)$ is trivial.

These results can be interpreted in the following way. Let M be a smooth closed manifold. There is a Poincaré duality map in K-theory:

$$K^0(TM) = K^0(DM, SM) \to K_0(DM) = K_0(M).$$

If one uses Atiyah's version of $K_0(M)$ [**1**], this map sends the symbol of an operator to the class of the operator considered as an element of $K_0(M)$. It follows from Theorem 5 that this Poincaré duality map depends on the differential structure.

These notions have been set in the framework of families of operators by A. Connes and G. Skandalis [**4**] in their work on index theory for foliated manifolds. They define a map $\psi^* \colon K^0(TM \times X) \to KK(M, X)$, which may be viewed as sending the symbol of a family of operators on M, parametrized by the compact space X, to the element of the Kasparov group [**7**] defined by that family. Again Theorem 5 implies that ψ^* depends on the differential structure on M. In this sense the index theorem for families is not topologically invariant, as opposed to the ordinary index theorem for a single operator.

REFERENCES

1. M. F. Atiyah, *Global theory of elliptic operators,* Proc. Internat. Conf. on Functional Analysis and Related Topics, Univ. of Tokyo Press, 1970.

2. L. G. Brown, R. G. Douglas and P. Fillmore, *Extensions of C^*-algebras and K-homology,* Ann. of Math. (2) **105** (1977), 265–324.

3. G. Brumfiel and I. Madsen, *Evaluation of the transfer and the universal surgery obstruction class,* Invent. Math. **32** (1976), 133–169.

4. A. Connes and G. Skandalis, *The longitudinal index theorem for foliations,* Publ. Res. Inst. Math. Science, Kyoto (to appear).

5. J. Kaminker, *Pseudo-differential operators and differential structures.* I, Contemporary Math., Vol. 10, Amer. Math. Soc., Providence, R.I., 1982, pp. 99–128.

6. ——, *Pseudo-differential operators and differential structures.* II. Amer. J. Math. (to appear).

7. G. G. Kasparov, *Operator K-functor and extensions of C^*-algebras,* Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 571–636.

8. I. M. Singer, *Some remarks on operator theory and index theory,* Proc. Conf. K-theory and Operator Algebras, Lecture Notes in Math, no. 575, Springer-Verlag, 1977, pp. 128–138.

9. D. Sullivan, *Geometric topology.* Part I: *Localization, periodicity, and Galois symmetry,* Mimeographed notes, M.I.T., 1970.

DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY-PURDUE UNIVERSITY, INDIANNAPOLIS, INDIANA 46223

Current address: Mathematical Science Research Institute, Berkeley, California 94720