BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

WEIGHTED NORM INEQUALITIES FOR POTENTIALS WITH APPLICATIONS TO SCHRÖDINGER OPERATORS, FOURIER TRANSFORMS, AND CARLESON MEASURES

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I. Introduction. A new characterization of the trace inequality for potential operators is given and used to sharpen recent results of C. L. Fefferman and D. H. Phong on the distribution of eigenvalues of Schrödinger operators. It is also used to study the domain and essential spectrum of Schrödinger operators, to obtain weighted norm inequalities for Fourier transforms, and to determine the Carleson measures for Dirichlet-type spaces.

THEOREM 1. *Suppose K is a nonnegative, locally integrable, radial function on R^n, which is decreasing as a function of $|x|$. For f in the class $P(R^n)$ of nonnegative, measurable functions on R^n and $x \in R^n$, set*

$$(Tf)(x) = (K * f)(x) = \int_{R^n} K(x-y) f(y)\, dy,$$

provided this integral exists for almost all $x \in R^n$. Then given $1 < p < \infty$ and $v \in P(R^n)$, there exists $C > 0$ so that the trace inequality

$$(1) \qquad \int_{R^n} (Tf)(x)^p v(x)\, dx \leq C \int_{R^n} f(x)^p\, dx, \qquad f \in P(R^n),$$

holds if and only if $C' > 0$ exists with

$$(2) \quad \int_Q T(\chi_Q v)(x)^{p'}\, dx \leq C' \int_Q v(x)\, dx < \infty \quad \text{for all dyadic cubes } Q,$$

where, as usual, $p' = p/(p-1)$.

Alternative characterizations of the trace inequality in terms of L^p capacities have been obtained in [**1** and **4**].

The trace inequality (1), for $p = 2$, and the potential kernel $K^\alpha(x)$, with $\hat{K}^\alpha(\varsigma) = (\alpha + |\varsigma|^2)^{-1/2}$, arises in estimating the eigenvalues[3] of a Schrödinger operator H. Let

$$(I_2 f)(x) = \int_{R^n} |x-y|^{2-n} f(y)\, dy$$

Received by the editors March 26, 1984 and, in revised form, August 15, 1984.
1980 *Mathematics Subject Classification*. Primary 26D10, 42B25.
Key words and phrases. Weighted norm inequality, potential operator, Schrödinger operator, Fourier transform, Carleson measure.
[1]Research supported in part by N.S.E.R.C. grant A4021.
[2]Research supported in part by N.S.E.R.C. grant A5149.

[3]By eigenvalues we mean the numbers $\lambda_1 < \cdots < \lambda_N < \cdots$, where λ_N is the maximum over all $N-1$ tuples $\phi_1, \ldots, \phi_{N-1}$ of the quantity $\inf \langle Hu, u\rangle / \langle u, u\rangle$, the infimum being over all $u \in Q(H)$, $u \perp \phi_j$, $j = 1, \ldots, N-1$. Here $Q(H)$ denotes the form domain of H. See [**10**].

denote the Newtonian potential of f. The following result refines the estimates of the least eigenvalue of H given in Theorem 5 of [**3**].

THEOREM 2. *Let $H = -\Delta - v$, where $v \in P(R^n), n \geq 3$. Denote the v measure of Q, $\int_Q v(x)\,dx$, by $|Q|_v$. There are positive constants C and c, depending only on the dimension n, such that the least eigenvalue, λ_1, of H satisfies $E_{\mathrm{sm}} \leq -\lambda_1 \leq E_{\mathrm{big}}$, where*

$$E_{\mathrm{sm}} = \sup\left\{|Q|^{-2/n} \colon |Q|_v^{-1}\int_Q I_2(\chi_Q v)v \geq C\right\},$$
$$E_{\mathrm{big}} = \sup\left\{|Q|^{-2/n} \colon |Q|_v^{-1}\int_Q I_2(\chi_Q v)v \geq c\right\}.$$

A similar refinement of Theorems 6 and 6′ in [**3**] is given in

THEOREM 3. *Let $H = -\Delta - v$, where $v \in P(R^n)$, $n \geq 3$. There are positive constants C and c, depending only on the dimension n, such that*

(A) *H has at least N eigenvalues $\leq -\lambda$, $\lambda > 0$, provided there exists a collection of N cubes $Q_1, \ldots, Q_N$ of side length at most $\lambda^{-1/2}$, whose doubles are pairwise disjoint, with $|Q_j|_v^{-1}\int_{Q_j} I_2(\chi_{Q_j} v)v \geq C$, $1 \leq j \leq N$.*

Conversely,

(B) *H having at least CN eigenvalues $\leq -\lambda$ implies there is a collection of N pairwise disjoint dyadic cubes $Q_1, \ldots, Q_N$, of side length at most $\lambda^{-1/2}$, that satisfy*

$$|Q_j|_v^{-1}\int_{Q_j} I_2(\chi_{Q_j} v)v \geq c, \qquad 1 \leq j \leq N.$$

REMARKS. 1. Roughly speaking, Theorem 3 says that the negative eigenvalues of H are approximately given by $-|Q|^{-2/n}$ as Q ranges over all the minimal dyadic cubes satisfying $|Q|_v^{-1}\int_Q I_2(\chi_Q v)v \geq C$.

2. As an illustration of Theorem 2, consider Example V in [**3**]: a particle in a rectangular box $B = B_1 \times B_2 \times \cdots \times B_n$ with side lengths $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_n$. Let $v = \chi_B$ and x_B denote the centre of B. Since

$$\begin{aligned}\sup_Q |Q|_v^{-1}\int_Q I_2(\chi_Q v)v &\cong I_2 v(x_B) \cong \delta_1^2 + \delta_1\delta_2(1 + \log\delta_3/\delta_2)\\ &\cong \delta_1\delta_2\log(1 + \delta_3/\delta_2),\end{aligned}$$

Theorem 2 yields the correct order of magnitude for the energy, E_{critical}, needed to trap a particle in B, namely

$$E_{\mathrm{critical}} = \sup\{E \geq 0\colon\ -\Delta - Ev \geq 0\} \cong (\delta_1\delta_2\log(1 + \delta_3/\delta_2))^{-1}.$$

3. The quantity $|Q|_v^{-1}\int_Q I_2(\chi_Q v)v$ is, in a sense, intermediate between the simpler ones used in [**3**] for the results corresponding to (A) and (B). Indeed, it is possible to show that for $p > 1$,

$$|Q|^{2/n-1}\int_Q v \le C|Q|_v^{-1}\int_Q I_2(\chi_Q v)v \le C_p \sup_{Q'\subset Q}|Q'|^{2/n-1/p}\left(\int_{Q'} v^p\right)^{1/p}.^{4}$$

The trace inequality also arises in questions concerning the domain and essential spectrum of Schrödinger operators. For example, conditions like (2) determine when the operator T in (1) is compact. This leads to conditions sufficient for H to have the same essential spectrum as $-\Delta$, that is, $[0,\infty)$.[4]

Another application of Theorem 1 is to weighted inequalities for Fourier transforms on R.

THEOREM 4. *Suppose $u(x)$ is an even, locally integrable function on R which is convex and decreases to 0 on $(0,\infty)$. Then for arbitrary $v(x)\ge 0$,*

$$(3)\qquad \int_{-\infty}^{\infty}|\hat f(x)u(x)|^2\,dx \le C\int_{-\infty}^{\infty}|f(x)v(x)|^2\,dx \quad \textit{for all } f\in L^1(R)$$

if and only if

$$\int_I \tilde M(\chi_I v^{-2})(x)^2\,dx \le C'\int_I v(x)^{-2}\,dx \quad \textit{for all intervals } I,$$

where

$$(\tilde M f)(x) = \sup_{x\in I}\left[\int_0^{|I|^{-1}} u(y)\,dy\right]\int_I |f(y)|\,dy.$$

For earlier conditions guaranteeing (3) see [**5**, **6**, and **7**].

Our final application is to Carleson measures for the Dirichlet-type spaces h_K^p introduced in [**9**]. The space h_K^p consists of the Poisson integrals, u, of potentials $K*f$, $f\in L^p(R^n)$. A positive measure μ on R_+^{n+1} is said to be a Carleson measure for h_K^p if $\|u\|_{L^p(\mu)}\le C\|f\|_p$ for all $f\in L^p(R^n)$.

THEOREM 5. *Suppose $K(x)$ is nonnegative and radial on R^n and is decreasing as a function of $|x|$. Then for $1<p<\infty$, a positive Borel measure μ on R_+^{n+1} is a Carleson measure for h_K^p if and only if*

$$\int_Q \overline{M}(\chi_{T(Q)}\mu)(x)^{p'}\,dx \le C\int_{T(Q)} d\mu < \infty \quad \textit{for all cubes } Q.$$

Here, Q is a cube in R^n and $T(Q)$ denotes the cube in R_+^{n+1} having Q as a face. The Carleson maximal function, $\overline{M}\nu$, is given at $x\in R^n$ by

$$\overline{M}\nu(x) = \sup_{x\in Q}\left[|Q|^{-1}\int_{|y|\le|Q|^{1/n}} K(y)\,dy\right]\int_{T(Q)} d\nu.$$

A characterization of Carleson measures in terms of L^p capacities can be found in [**9** and **12**].

[4] We wish to thank M. Wilson for communicating to us an alternative proof of the connection between Theorem 1 and the results of [**3**].

II. Sketch of proofs.

PROOF OF THEOREM 1 By duality, (1) is equivalent to

$$\int T(gv)^{p'} \leq C' \int g^{p'} v, \qquad g \in P(R^n). \tag{4}$$

Extensions of theorems in [**8**] show that (4) amounts to the same inequality with T replaced by the dyadic maximal operator

$$(Mf)(x) = \sup_{x \in Q} \left[|Q|^{-1} \int_{|y| \leq |Q|^{1/n}} K(y)\, dy \right] \int_Q |f|, \qquad Q \text{ dyadic.}$$

The methods of [**11**] now yield (2), with M instead of T, as necessary and sufficient for the latter inequality. Finally, as above, M and T are interchangeable, so the proof is complete.

PROOF OF THEOREM 2. We have

$$-\lambda_1 \equiv \sup_{u \in Q(H)} -\frac{\langle Hu, u\rangle}{\langle u, u\rangle} = \inf\{\alpha > 0 \colon C_\alpha \leq 1\},$$

where C_α is the least constant such that $\int_{R^n} (I_1^\alpha f)^2 v \leq C_\alpha \int f^2$ for all $f \in P(R^n)$, and I_k^α has kernel K_k^α with $\hat{K}_k^\alpha(\varsigma) = (\alpha + |\varsigma|^2)^{-k/2}$. This is so since

$$\begin{aligned}
\sup_{u \in Q(H)} -\frac{\langle Hu, u\rangle}{\langle u, u\rangle} &= \sup_{u \in Q(H)} \frac{\int |u|^2 v - \int |\nabla u|^2}{\int |u|^2} \\
&= \inf\left\{\alpha > 0 \colon \int |u|^2 v \leq \int (\alpha |u|^2 + |\nabla u|^2) = \int (\alpha + |\varsigma|^2) |\hat{u}(\varsigma)|^2\, d\varsigma \right\} \\
&= \inf\{\alpha > 0 \colon C_\alpha \leq 1\}.
\end{aligned}$$

Theorem 1 now yields $C_\alpha \cong \sup_Q |Q|_v^{-1} \int I_1^\alpha (\chi_Q v)^2$. Standard estimates on Bessel kernels show it suffices to take this supremum over cubes of side length at most $\alpha^{-1/2}$, so Theorem 2 follows readily, since

$$\int I_1^\alpha (\chi_Q v)^2 = \int_Q I_2^\alpha (\chi_Q v) v \cong \int_Q I_2 (\chi_Q v) v$$

for such cubes.

PROOF OF THEOREM 3. (A) As in [**3**], it suffices to construct an N-dimensional subspace S of $Q(H)$ such that $\int (|\nabla u|^2 + \lambda |u|^2) \leq \int |u|^2 v$, $u \in S$. With some computation, one verifies this inequality for

$$S = \operatorname{Span}\{\theta_j I_2^\lambda (\chi_{Q_j} v)\}_{j=1}^N,$$

the θ_j being dilates and translates of a fixed C^∞ function θ with $\theta_j \equiv 1$ on $\frac{3}{2} Q_j$, $\operatorname{supp} \theta_j \subset 2Q_j$, $j = 1, \ldots, N$.

(B) We sketch the case $\lambda = 0$, following the line of proof in [**3**]. Thus, we prove (B) by showing that if $Q_1, \ldots, Q_N$ are all the minimal dyadic cubes satisfying $|Q|_v^{-1} \int_Q I_2(\chi_Q v) v \geq c$, then H has at most CN negative eigenvalues. This is done by constructing a subspace S of codimension CN in L^2 such that $\int |u|^2 v \leq \int |\nabla u|^2$, $u \in S \cap Q(H)$. We define additional cubes $Q_{N+1}, \ldots, Q_M$, $M \leq CN$, and sets E_j, $0 \leq j \leq M$, in analogy with those in [**3**]. A modification of arguments in [**3**] shows that if

$v_j = \chi_{E_j} v$, then $|Q|_{v_j}^{-1} \int_Q I_2(\chi_Q v_j) v_j \le c$ for all dyadic cubes Q and, thus, $\int (I_1 f)^2 v_j \le \int f^2$, $f \in P(R^n)$, $0 \le j \le M$, by Theorem 1.

It is possible to find cubes Q_j^i (not necessarily dyadic or pairwise disjoint) such that $\bigcup_i Q_j^i = E_j$ for $0 \le j \le M$ and such that the total number of Q_j^i does not exceed $C_n M$. Let $S = \{u \in L^2 \colon \int_{Q_j^i} u = 0 \text{ for all } i, j\}$. Lemma 1.4 of [**2**] shows

$$|u(x)| \le C I_1(\chi_{Q_j^i} |\nabla u|)(x) \le C I_1(\chi_{E_j} |\nabla u|)(x)$$

for $x \in Q_j^i, u \in S$, and so

$$\begin{aligned} \int |u|^2 v &= \sum_{j=0}^{M} \int |u|^2 v_j \le C \sum_{j=0}^{M} \int [I_1(\chi_{E_j} |\nabla u|)]^2 v_j \\ &\le \sum_{j=0}^{M} \int_{E_j} |\nabla u|^2 = \int |\nabla u|^2 \end{aligned}$$

for $u \in S \cap Q(H)$, as required.

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