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A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]

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A theorem of J. Alexander [**1**] asserts that any tame oriented link in 3-space may be represented by a pair (b, n), where b is an element of the n-string braid group B_n. The link L is obtained by closing b, i.e., tying the top end of each string to the same position on the bottom of the braid as shown in Figure 1. The closed braid will be denoted $b^\wedge$.

Thus, the trivial link with n components is represented by the pair $(1, n)$, and the unknot is represented by $(s_1 s_2 \cdots s_{n-1}, n)$ for any n, where $s_1, s_2, \ldots, s_{n-1}$ are the usual generators for B_n.

The second example shows that the correspondence of (b, n) with $b^\wedge$ is many-to-one, and a theorem of A. Markov [**15**] answers, in theory, the question of when two braids represent the same link. A Markov move of type 1 is the replacement of (b, n) by (gbg^{-1}, n) for any element g in B_n, and a Markov move of type 2 is the replacement of (b, n) by $(bs_n^{\pm 1}, n+1)$. Markov's theorem asserts that (b, n) and (c, m) represent the same closed braid (up to link isotopy) if and only if they are equivalent for the equivalence relation generated by Markov moves of types 1 and 2 on the *disjoint* union of the braid groups. Unforunately, although the conjugacy problem has been solved by F. Garside [**8**] within each braid group, there is no known algorithm to decide when (b, n) and (c, m) are equivalent. For a proof of Markov's theorem see J. Birman's book [**4**].

The difficulty of applying Markov's theorem has made it difficult to use braids to study links. The main evidence that they might be useful was the existence of a representation of dimension $n-1$ of B_n discovered by W. Burau in [**5**]. The representation has a parameter t, and it turns out that the determinant of $1-$(Burau matrix) gives the Alexander polynomial of the closed braid. Even so, the Alexander polynomial occurs with a normalization which seemed difficult to predict.

In this note we introduce a polynomial invariant for tame oriented links via certain representations of the braid group. That the invariant depends only on the closed braid is a direct consequence of Markov's theorem and a certain trace formula, which was discovered because of the uniqueness of the trace on certain von Neumann algebras called type II_1 factors.

Notation. In this paper the Alexander polynomial Δ will always be normalized so that it is symmetric in t and t^{-1} and satisfies $\Delta(1) = 1$ as in Conway's tables in [**6**].

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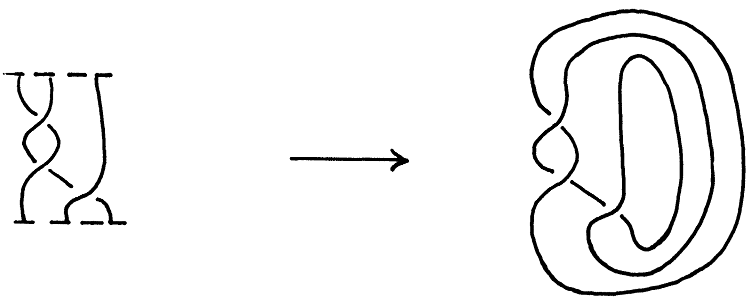


b $b^\wedge$

FIGURE 1

While investigating the index of a subfactor of a type II_1 factor, the author was led to analyze certain *finite-dimensional* von Neumann algebras A_n generated by an identity 1 and n projections, which we shall call $e_1, e_2, \ldots, e_n$. They satisfy the relations

(I) $e_i^2 = e_i, e_i^* = e_i$,

(II) $e_i e_{i\pm 1} e_i = t/(1+t)^2 e_i$,

(III) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.

Here t is a complex number. It has been shown by H. Wenzl [**24**] that an arbitrarily large family of such projections can only exist if t is either real and positive or $e^{\pm 2\pi i/k}$ for some $k = 3, 4, 5, \ldots$. When t is one of these numbers, there exists such an algebra for all n possessing a trace $\mathrm{tr}\colon A_n \to \mathbf{C}$ completely determined by the normalization $\mathrm{tr}(1) = 1$ and

(IV) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,

(V) $\mathrm{tr}(we_{n+1}) = t/(1+t)^2 \,\mathrm{tr}(w)$ if w is in A_n,

(VI) $\mathrm{tr}(a^* a) > 0$ if $a \neq 0$

(note $A_0 = \mathbf{C}$).

Conditions (I)–(VI) determine the structure of A up to $*$-isomorphism. This fact was proved in [**9**], and a more detailed description appears in [**10**]. Remember that a finite-dimensional von Neumann algebra is just a product of matrix algebras, the $*$ operation being conjugate-transpose.

For real t, D. Evans pointed out that an explicit representation of A_n on $\mathbf{C}^{2n+2}$ was discovered by H. Temperley and E. Lieb [**23**], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A readable account of this can be found in R. Baxter's book [**2**]. This representation was rediscovered in the von Neumann algebra context by M. Pimsner and S. Popa [**18**], who also found that the trace tr is given by the restriction of the Powers state with $t = \lambda$ (see [**18**]).

For the roots of unity the algebras A_n are intimately connected with Coxeter groups in a way that is far from understood.

The similarity between relations (II) and (III) and Artin's presentation of the n-string braid group,

$$\{s_1, s_2, \ldots, s_n\colon s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \text{ if } |i-j| \geq 2\},$$

was first pointed out by D. Hatt and P. de la Harpe. It transpires that if one defines $g_i = \sqrt{t}(te_i - (1 - e_i))$, the g_i satisfy the correct relations, and one obtains representations r_t of B_n by sending s_i to g_i.

THEOREM 1. *The number* $(-(t+1)/\sqrt{t})^{n-1}\operatorname{tr}(r_t(b))$ *for* b *in* B_n *depends only on the isotopy class of the closed braid* $b^\wedge$.

DEFINITION. If L is a tame oriented classical link, the *trace invariant* $V_L(t)$ is defined by

$$V_L(t) = (-(t+1)/\sqrt{t})^{n-1}\operatorname{tr}(r_t(b))$$

for any (b, n) such that $b^\wedge = L$.

The Hecke algebra approach shows the following.

THEOREM 2. *If the link* L *has an odd number of components,* $V_L(t)$ *is a Laurent polynomial over the integers. If the number of components is even,* $V_L(t)$ *is* $\sqrt{t}$ *times a Laurent polynomial.*

The reader may have observed that the von Neumann algebra structure (i.e., the $*$ operation) and condition (VI) are redundant for the definition of $V_L(t)$. This explains why V_L can be extended to all values of t except 0. However, it must be pointed out that for positive t and the relevant roots of unity, the presence of *positivity* gives a powerful method of proof.

The trace invariant depends on the oriented link but not on the chosen orientation. Let $L^\sim$ denote the mirror image link of L.

THEOREM 3. $V_{L^\sim}(t) = V_L(1/t)$.

Thus, the trace invariant can be used to detect a lack of amphicheirality. It seems to be very good at this. A glance at Table 1 shows that it distinguishes the trefoil knot from its mirror image and hence, via Theorem 6, it distinguishes the two granny knots and the square knot.

CONJECTURE 4. If L is not amphicheiral, $V_{L^\sim} \neq V_L$.

There is some evidence for this conjecture, but only \$10 hangs on it. In this direction we have the following result, where b is in B_n, b_+ is the sum of the positive exponents of b, and b_- is the (unsigned) sum of the negative ones in some expression for b as a word on the usual generators.

THEOREM 5. *If* $b_+ - 3b_- - n + 1$ *is positive, then* $b^\wedge$ *is not amphicheiral.*

For $b_- = 0$, i.e., positive braids, this follows from a recent result of L. Rudolf [**21**]. Also, if the condition of the theorem holds, we conclude that $b^\wedge$ is not the unknot. This is similar in kind to a recent result of D. Bennequin [**3**].

The connected sum of two links can be handled in the braid group provided one pays proper attention to the components being joined. Let us ignore the subtleties and state the following (where # denotes the connected sum).

THEOREM 6. $V_{L_1 \# L_2} = V_{L_1} V_{L_2}$.

As evidence for the power of the trace invariant, let us answer two questions posed in [**4**]. Both proofs are motivated by the fact, shown in [**10**], that $r_t(B_n)$ is sometimes finite.

THEOREM 7. *For every n there are infinitely many words in B_{n+1} which give close braids inequivalent to closed braids coming from elements of the form $Us_n^{-1}Vs_n$, where U and V are in B_n.*

Explicit examples are easy to find; e.g., all but a finite number of powers of $s_1^{-1}s_2s_3$ will do.

THEOREM 8 (SEE [**4** P. 217, Q. 8]). *If b is in B_n and there is an integer k greater than 3 for which $b \in \ker r_t$, $t = e^{2\pi\imath/k}$, then $b^\wedge$ has braid index n.*

Here the braid index of a link L is the smallest n for which there is a pair (b, n) with $b^\wedge = L$. The kernel of r_t is not hard to get into for these values of t.

COROLLARY 9. *If the greatest common divisor of the exponents of $b \in B_n$ is more than 1, then the braid index of $b^\wedge$ is n.*

More interesting examples can be obtained by using generators and relations for certain finite groups; e.g., the finite simple group of order 25,920 (see [**10**, **7**]). In general, the trace invariant can probably be used to determine the braid index in a great many cases.

Note also that the trace invariant detects the kernel of r_t.

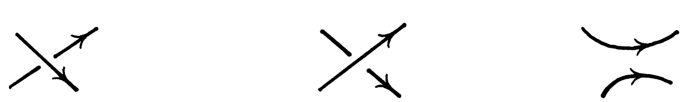
THEOREM 10. *For $t = e^{2\pi\imath/k}, k = 3, 4, 5, \ldots$, $V_{b^\wedge}(t) = (-2\cos\pi/k)^{n-1}$ if and only if $b \in \ker r_t$ (for $b \in B_n$).*

COROLLARY 11. *For transcendental t, $b \in \ker r_t$ if and only if $V_{b^\wedge}(t) = (-(t+1)/\sqrt{t})^{n-1}$.*

For transcendental t, r_t is very likely to be faithful.

There is an alternate way to calculate V_L without first converting L into a closed braid. In [**6**] Conway describes a method for rapidly computing the Alexander polynomials of links inductively. In fact, his first identity suffices in principle—see [**11**]. This identity is as follows.

Let L^+, L^-, and L be links related as in Figure 2, the rest of the links being identical. Then $\Delta_{L^+} - \Delta_{L^-} = (\sqrt{t} - 1/\sqrt{t})\Delta_L$.



L^+ L^- L

FIGURE 2

For the trace invariant we have

THEOREM 12. $1/tV_{L^-} - tV_{L^+} = (\sqrt{t} - 1/\sqrt{t})V_L$.

COROLLARY 13. *For any link* L, $V_L(-1) = \Delta_L(-1)$.

That the trace invariant may always be calculated by using Theorem 12 follows from the proof of the same thing for the Alexander polynomial. We urge the reader to try this method on, say, the trefoil knot.

The special nature of the algebras A_n when t is a relevant root of unity can be exploited to give information about V_L at these values.

THEOREM 14. *If* K *is a knot then* $V_K(e^{2\pi i/3}) = 1$.

THEOREM 15. $V_L(1) = (-2)^{p-1}$, *where* p *is the number of components of* L.

A more subtle analysis at $t = 1$ via the Temperley-Lieb-Pimsner-Popa representation gives the next result.

THEOREM 16. *If* K *is a knot then* $d/dtV_K(1) = 0$.

It is thus sensible to simplify the trace invariant for knots as follows.

DEFINITION 17. If K is a knot, define W_K to be the Laurent polynomial

$$W_K(t) = (1 - V_K(t))/(1 - t^3)(1 - t).$$

Amphicheirality is less obvious for W. In fact, $W_{K^\sim}(t) = 1/t^4 W_K(1/t)$. It is amusing that for W the unknot is 0 and the trefoil is 1. The connected sum is also less easy to see in the W picture. For the record the formula is

$$W_{K_1 \# K_2} = W_{K_1} + W_{K_2} - (1 - t)(1 - t^3)W_{K_1}W_{K_2}.$$

COROLLARY 18. $\Delta_K(-1) \equiv 1$ *or* $5 \pmod 8$.

When $t = i$ the algebras A_n are the complex Clifford algebras. This together with a recent result of J. Lannes [**13**] allows one to show the following.

THEOREM 19. *If* K *is a knot the Arf invariant is of* K *is* $W_K(i)$.

COROLLARY 20. $\Delta_K(-1) = 1$ *or* $5 \pmod 8$ *when the Arf invariant is* 0 *or* 1, *respectively.*

This is an alternate proof of a result in Levine [**14**]; also see [**11**, p. 155]. Note also that Corollary 20 allows one to define an Arf invariant for links as $V(i)$. It may be zero and is always plus or minus a power of two otherwise.

The values of V at $e^{\pi i/3}$ are also of considerable interest, as the algebra A_n is then related to a kind of cubic Clifford algebra. Also, in this case, $r_t(B_n)$ is always a finite group, so one can obtain a rapid method for calculating $V(t)$ without knowing V completely. We have included this value of V in the tables. Note that it is always in $1 + 2\mathbf{Z}(e^{i\pi/3})$.

There is yet a third way to calculate the trace invariant. The decomposition of A_n as a direct sum of matrix algebras is known [**10**], and H. Wenzl has explicit formulae for the (irreducible) representations of the braid group in each direct summand. So in principle this method could always be used. This brings in the Burau representation as a direct summand of r_t. For 3 and 4 braids this allows one to deduce some powerful relations with the

Alexander polynomial. An application of Theorem 16 allows one to determine the normalization of the Alexander polynomial in the Burau matrix for proper knots, and one has the following formulae.

THEOREM 21. *If b in B_3 has exponent sum e, and $b^\wedge$ is a knot, then*

$$V_{b^\wedge}(t) = t^{e/2}(1 + t^e + t + 1/t - t^{e/2-1}(1 + t + t^2)\Delta_{b^\wedge}(t)).$$

THEOREM 22. *If b in B_4 has exponent sum e, and $b^\wedge$ is a knot, then*

$$\begin{aligned} t^{-e}V(t) + t^e V(1/t) =&(t^{-3/2} + t^{-1/2} + t^{1/2} + t^{3/2})(t^{e/2} + t^{-e/2}) \\ &- (t^{-2} + t^{-1} + 2 + t + t^2)\Delta(t) \end{aligned}$$

(*where* $V = V_{b^\wedge}$ *and* $\Delta = \Delta_{b^\wedge}$).

These formulas have many interesting consequences. They show that, except in special cases, e is a knot invariant. They also give many obstructions to being closed 3 and 4 braids.

COROLLARY 23. *If K is a knot and $|\Delta_K(i)| > 3$, then K cannot be represented as a closed* 3 *braid.*

Of the 59 knots with 9 crossings or less which are known not to be closed 3 braids, this simple criterion establishes the result for 43 of them, at a glance.

COROLLARY 24. *If K is a knot and $\Delta_K(e^{2\pi i/5}) > 6.5$, then K cannot be represented as a closed* 4 *braid.*

For $n > 4$ there should be no simple relation with the Alexander polynomial, since the other direct summands of r_t look less and less like Burau representations.

In conclusion, we would like to point out that the q-state Potts model could be solved if one understood enough about the trace invariant for braids resembling certain braids discovered by sailors and known variously as the "French sinnet" (sennit) or the "tresse anglaise", depending on the nationality of the sailor. See [**21**, p. 90].

The author would like to thank Joan Birman. It was because of a long discussion with her that the relation between condition (V) and Markov's theorem became clear.

Tables. A single example should serve to explain how to read the tables. The knot 8_8 has trace invariant

$$t^{-3}(-1 + 2t - 3t^2 + 5t^3 - 4t^4 + 4t^5 - 3t^6 + 2t^7 - t^8).$$

Its W invariant is

$$t^{-3}(1 - t + 2t^2 - t^3 + t^4).$$

A braid representation for it is

$$s_1^{-1}s_2s_1^2s_3^{-1}s_2^2s_3^{-2} \quad \text{in } B_4.$$

Also note that $w = e^{\pi i/3}$.

ADDED IN PROOF. The similarity between the relation of Theorem 12 and Conway's relation has led several authors to a two-variable generalization of V_L. This has been done (independently) by Lickorish and Millett, Ocneanu, Freyd and Yetter, and Hoste.

TABLE 1. The trace invariant for prime knots to 8 crossings.

knot	braid rep.	p_0	pol(V)	V(w)	p_0	pol(W)
0_1	1	0	1	1	0	0
3_1	1^3	1	101-1	$i\sqrt{3}$	0	1
4_1	$12^{-1}12^{-1}$	-2	1-11-11	-1	-2	-1
5_1	1^5	2	101-11-1	-1	0	1101
5_2	$2^21^{-1}21^2$	1	1-12-11-1	-1	0	101
6_1	$12^{-1}13^{-1}23^{-1}2^{-1}$	-4	1-11-22-11	$i\sqrt{3}$	-4	-10-1
6_2	$1^{-1}21^{-1}2^3$	-1	1-12-22-21	1	-1	-11-1
6_3	$1^{-1}2^21^{-2}2$	-3	-12-23-22-1	1	-3	1-11
7_1	1^7	3	101-11-11-1	-1	0	1111101
7_2	$1^{-1}3^321^23^{-1}2$	1	1-12-22-11-1	1	0	10101
7_3	$2^512^21^{-2}$	2	1-12-23-21-1	1	0	110201
7_4	$3^21^{-1}23^{-1}21^22$	1	1-23-23-21-1	-1	0	10201
7_5	$2^312^41^{-2}$	2	1-13-33-32-1	-1	0	1102-11
7_6	$3^{-1}1^{-1}2113^{-1}2^{-3}$	-6	-12-34-33-21	-1	-6	1-12-1
7_7	$13^{-1}23^{-1}21^{-1}23^{-1}2$	-3	-13-34-43-21	$-i\sqrt{3}$	-3	1-21-1
8_1	$12^{-1}3^{-1}214^{-2}.$ $3^{-1}2^{-1}4$	-6	1-11-22-22-11	1	-6	-10-10-1
8_2	$2^51^{-1}21^{-1}$	0	1-12-23-32-21	-1	1	1-11-1
8_3	$123^{-1}4^{-1}3^{-1}2$ $1^{-1}3^2243^{-1}2^{-2}$	-4	1-12-33-32-11	-3	-4	-10-20-1
8_4	$11132^{-1}3^{-2}12^{-1}$	-3	1-12-33-33-21	-1	-3	-10-21-1
8_5	$1112^{-1}1112^{-1}$	0	1-13-33-43-21	$i\sqrt{3}$	1	1-21-1
8_6	$3^{-2}12^{-1}132^{-3}$	-7	1-23-44-43-11	1	-7	-11-21-1
8_7	$2^{-2}12^{-1}1^4$	-2	-12-24-44-32-1	1	-2	1-12-11
8_8	$1^{-1}2113^{-1}223^{-2}$	-3	-12-35-44-32-1	1	-3	1-12-11
8_9	$2^31^{-1}21^{-3}$	-4	1-23-45-43-21	1	-4	-11-21-1
8_{10}	$221^{-2}2^31^{-1}$	-2	-12-35-45-42-1	$i\sqrt{3}$	-2	1-13-11
8_{11}	$12^{-2}32^{-1}3^{-2}12^{-1}$	-7	1-23-55-44-21	$-i\sqrt{3}$	-7	-11-22-1
8_{12}	$12^{-1}34^{-1}34^{-1}.$ $213^{-1}2^{-1}$	-4	1-24-55-54-21	-1	-4	-11-31-1
8_{13}	$1123^{-1}21^{-1}3^{-2}2$	-3	-13-45-55-32-1	-1	-3	1-22-11
8_{14}	$11221^{-1}3^{-1}23^{-1}2$	-1	1-24-56-54-31	-1	-1	-12-22-1
8_{15}	$111231^{-1}2^332^{-1}$	2	1-25-56-64-31	$-i\sqrt{3}$	0	1103-22-1
8_{16}	$112^{-1}112^{-1}12^{-1}$	-2	-13-46-66-53-1	1	-2	1-23-21
8_{17}	$21^{-1}21^{-1}221^{-2}$	-4	1-35-67-65-31	1	-4	-12-32-1
8_{18}	$(12^{-1})^3$	-4	1-46-79-76-41	3	-4	-13-33-1
8_{19}	121212^21	3	10100-1	$-i\sqrt{3}$	0	11111
8_{20}	21^321^{-3}	-1	-12-12-11-1	$i\sqrt{3}$	-1	101
8_{21}	$2^312^21^{-2}21^{-1}$	1	2-23-32-21	$i\sqrt{3}$	0	1-11-1

TABLE 2. The trace invariant for some divers knots and links.

link	p_0	pol(V)	V(w)	p_0	pol(W)	braid rep.
10_{141} (1)	-2	1-23-34-32-21	$i\sqrt{3}$	-2	-11-11-1	$2^{-4}112221$
KT (2)	-4	-12-22001-22-21	1	-4	1-110-11-1	$1113323^{-1}1^{-2}2.1^{-1}3^{-1}2^{-1}$
C (3)	-4	-12-22001-22-21	1	-4	1-110-11-1	$22213^{-1}2^{-2}1.2^{-1}13^{-1}$
2^2_1	1/2	-10-1	-i			1^2
4^2_1 (4)	3/2	-10-11-1	-i			1^4
4^2_1 (4)	1/2	-11-10-1	i			$12^{-1}122$
5^2_1	-7/2	1-21-21-1	i			$12^{-1}12^{-2}$
6^2_1	5/2	-10-11-11-1	$\sqrt{3}$			1^6
6^2_2	3/2	-11-22-21-1	-i			2221121^{-1}
6^2_3	-3/2	-12-22-31-1	$\sqrt{3}$			$21^{-1}23^{-1}2123^{-1}$
H_1 (5)	1/2	-11-10-1	i			$12^{-1}122$
H_2 (5)	-3/2	1-10-10-1	-i			$12^{-3}122$
W (6)	-3/2	-11-21-21	-i			$11221^{-1}2^{-2}$
6^3_1	-1	1-13-13-21	$i\sqrt{3}$			$221^{-1}221^{-1}$
6^3_2	-3	-13-24-23-1	1			$12^{-1}12^{-1}12^{-1}$
6^3_3	2	10102	1			122122
A (7)	5/2	-10-32-34-22-1	3i			11222333
B (7)	5/2	-10-32-34-22-1	3i			11122333

Table Notes

(1) Compare 8_5 which has the same Alexander polynomial.
(2) The Kinoshita-Terasaka knot with 11 crossings. See [**12**].
(3) This is Conway's knot with trivia Alexander polynomial. See [**20**].
(4) Same link, different orientation.
(5) These links have homeomorphic complements.
(6) The Whitehead link.
(7) Two composite links with the same trace invariant.

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