J. Hâjek (1972), *Local asymptotic minimax and admissibility in estimation*, Proc. Sixth Berkeley Sympos. Math. Statist. and Probab. Vol. 1, pp. 175–194.

J. Hâjek and Z. Šidâk (1967), *Theory of rank tests*, Academia, Praha.

P. Hall and C. C. Heyde (1980), *Martingale limit theory and its application*, Academic Press, New York.

P. Jeganathan (1980), *Asymptotic theory of estimation when the limit of the loglikelihood ratios is mixed normal*, Thesis, Indian Statist. Inst. Calcutta.

Yu. A. Koshevnik and B. Ya. Levit (1976), *On a non-parametric analogue of the information matrix*, Theory Probab. Appl. **21**, 738–753.

P. Laplace (1809–1810), *Approximation des formules qui sont fonctions de très grands nombres et leur application aux probabilitiés*, Mémoires de l'Académie des Sciences, vol. 10.

L. Le Cam (1960), *Locally asymptotically normal families of distributions*, Univ. California Publ. Statist., vol. 3, pp. 37–98.

E. L. Lehmann (1975), *Nonparametrics. Statistical methods based on ranks*, Holden-Day, San Francisco, Calif.

P. W. Millar (1979), *Asymptotic minimax theorems for the sample distribution function*, Z. Wahrsch. Verw. Gebiete **48**, 233–252.

______ (1983), *Nonparametric applications of an infinite dimensional convolution theorem*, Preprint, Dept. Statist., Berkeley.

W. Moussatat (1976), *On the asymptotic theory of statistical experiments and some of its applications*, Thesis, Univ. of California, Berkeley.

J. Pfanzagl (1980), *Asymptotic expansions in parametric statistical theory*, Developments in Statistics (P. R. Krishnaiah, ed.), Academic Press, New York, pp. 1–97.

R. Serfling (1982), *Approximation theorems of mathematical statistics*, Wiley, New York,

S. Saks (1937), *Theory of the integral*, 2nd ed., Hafner, New York.

C. Stein (1956), *Efficient nonparametric testing and estimation*, Proc. Third Berkeley Sympos. Math. Statist. and Probab., Vol. 1, pp. 187–195.

A. Wald (1943), *Tests of statistical hypotheses concerning several parameters when the number of observations is large*, Trans. Amer. Math. Soc. **54**, 426–482.

LUCIEN LE CAM

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 11, Number 2, October 1984

0273-0979/84 $1.00 + $.25 per page

Manifolds all of whose geodesics are closed, by Arthur L. Besse, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 93, Springer-Verlag, Berlin-Heidelberg-New York, ix + 262 pp., 1978, $40.00. ISBN 0-3870-8158-5

In differential geometry, as in many branches of mathematics, the practitioners can be classified roughly into two groups, the structuralists and the problem solvers. Flourishing in the time of Hilbert and reaching a peak sometime after the appearance of the Bourbaki series, the structuralists gained the upper hand. However, the titles of books recently published, such as *Comparison theorems in Riemannian geometry* by J. Cheeger and D. Ebin, North-Holland, 1975, and the book at hand seem to indicate that the rococo in mathematics, especially differential geometry, has come back in force.

If we probe a little deeper, we find that the relation between the two schools is more cooperative than competitive. Some of the problems discussed in this book, including the central one, Blaschke's problem of analyzing "Wiedersehensflächen," were posed more than 50 years ago. In terms of the mathematics of that day there was little chance of framing a satisfactory solution. The intervening period of structural dominance has produced a broad base which makes it possible and natural to formulate the solve sweeping generalizations of the original problems.

As soon as we get beyond the title page we find that the book is not so special after all. The idea of studying manifolds with many closed geodesics is only a guiding thread. Much space is devoted to tangential issues, such as a full chapter (out of 9) on harmonic manifolds. The author bears out the quip that differential geometry is the study of objects invariant under change of notation by starting with an unusual foundational chapter. The feature which makes it unusual is an emphasis on symplectic structures. Let us examine briefly the traditional description of Riemannian geometry and how it has evolved with our understanding of structure. It should become clear why the link with symplectic structures is apt for the kinds of problems Besse aims to study.

The classical approach to basic Riemannian geometry is in terms of coordinates. Geodesics were originally conceived as distance-minimizing curves and thus were realized as the solutions of a system of second-order differential equations. It was Levi-Civita's achievement to put these differential equations in a broader context, seeing them as the generalization of the fact that the tangent vectors to a straight line are parallel. Thus, he was led to invent a notion of vectors in parallel motion along a curve; the rate of deviation from parallel translation is called the covariant derivative. At that time mathematicians were largely content to be able to operate with objects in terms of coordinate-dependent indexed arrays. They knew what they were doing applied to many notable examples, and it was left to later generations to abstract and specify the context in which it all worked. Thus it was not until the 1930s that Whitney established that the extrinsic and intrinsic ideas of a manifold were both equivalent to his abstraction. E. Cartan seemed to think consistently in terms of prolongations of basic spaces to others which carry geometrical structure, but it was the later generation of Chern, Ehresman, and Steenrod (to mention a few) who felt the need to formulate the notion of fibre bundle and develop methods to distinguish between them [**4**]. Now this bundle-structure game has been stirring excitement among physicists because it gives an elegant setting for Yang-Mills gauge theory. A companion to Riemannian structures, linked by the calculus of variations, is the mathematics of mechanics [**1, 2, 5, 6, 7**]. The tie is preserved in their bundle-theoretic formulations. Thus in mechanics we have a phase space, having as its global form the cotangent bundle, on which the canonical 2-form is defined and gives it a symplectic structure. A total-energy function on this space leads to the Hamilton-Jacobi equations, which are interpreted as a vector field. The transformation from the Lagrangian approach, and hence the link with calculus of variations, is accomplished by fiber derivatives. Riemannian geometry is the special case for which the total energy is the fundamental quadratic form, conceived of as a real-valued function on the tangent bundle by the "musical isomorphism". (This is Besse's name for it, with sharps and flats for notation. From [**3**, p. 197] we have "la valse des indices co- et contravariants sous la baguette du tenseur metrique.") Via this isomorphism the Hamilton-Jacobi vector field is brought to the tangent bundle, where it becomes the geodesic vector field. To present-day mathematicians, trained to think operationally and chase diagrams, this procedure gives a clear and satisfying big picture.

The link between Riemannian and symplectic structures can be taken a step further, because locally the space of unit-speed geodesics is itself a symplectic manifold. The tangent vectors to that space are the vector fields along the geodesics induced by a variation which moves geodesics through geodesics without translating or stretching; that is, the normal Jacobi fields. It is well known that if X and Y are Jacobi fields on a geodesic, then $g(X, Y') - g(Y, X')$ is constant, where g is the metric tensor and prime indicates covariant derivative in the direction of the geodesic. We denote this constant by $\omega(X, Y)$. Obviously ω is a 2-form on the tangent space to the space of geodesics. It is not hard to see that it is nondegenerate and closed. In fact it comes from the symplectic form on the tangent bundle by the processes of restricting to the unit tangent bundle and quotienting with respect to the geodesic flow. Now the notions of symplectic geometry gain a special significance. Particularly, the Lagrangian submanifolds are the submanifolds which spray normally from a hypersurface N; the tangent space to such a Lagrangian submanifold is the space of N-Jacobi fields. The geodesics spraying normally from a lower-dimensional submanifold P also form a Lagrangian submanifold; in fact, we can take as hypersurface N a tube about P. However, generally a Lagrangian submanifold does not focus on such a P; indeed, the singularities which arise are an interesting object of study.

To obtain a suitable global theory for the space of geodesics requires additional properties. By way of noting how bad it can be, recall the Hadamard theorems on compact surfaces of negative curvature: given an ordered set of k geodesic segments and an $\varepsilon > 0$, there is a single geodesic which sweeps by the k segments in turn ε-close. The modern generalization is the Arnol'd theorem that the geodesic flow is ergodic. Obviously with such a hypothesis the set of geodesics is extremely unlike a manifold globally. On the other hand, a very powerful assumption is made by Besse in his discussion of the space of geodesics, namely, that all the geodesics are closed and of the same period. Then the space of geodesics is not only a manifold, but also compact. For this case the symplectic structure has been used by A. Weinstein [**8**] to prove that the volume of the original manifold is an integral multiple of the volume of a sphere having the same geodesic period.

The description of the known examples of manifolds having all geodesics closed and of the same length involves a large variety of geometrical structures. The homogeneous ones are the compact rank-one symmetric spaces (CROSSes), which are, explicitly, spheres and various projective spaces. Besse gives a concise introduction to the several ways their study can be approached. There is the differential geometry approach, for which one can start with the basic assumption that the curvature tensor is parallel along all curves. Then the condition that it have rank one is given by the requirement that there be no totally geodesic flat submanifolds except geodesics. The same spaces can be specified by giving the groups which act transitively and the isotropy subgroups. Their rich collection of totally geodesic submanifolds admits axiomatic characterization as special sorts of classical projective spaces. In particular, the intricate distinction between Arguesian geometries and Moufang planes illuminates the fact that over the reals, complexes, and quaternions we have spaces of all dimensions, but there is only a plane over the Cayley algebra. Their topological invariants can be understood either analytically via de Rham's Theorem or by cellular decompositions produced out of their projective-space structures.

The nonhomogeneous surface examples were already studied by Darboux and his contemporaries, Tannery and Zoll. They are surfaces which admit a circle group of isometries, and so can be locally isometrically embedded as patches on surfaces of revolution. Some are globally surfaces of revolution. The theory of symplectic manifolds gives an explanation of why it is possible to calculate geodesics explicitly enough to sort out the ones with closed geodesics; namely, when there is a symmetry group there are associated first integrals of the potential system. This led A. Weinstein to try for and get higher dimensional examples on S^n which have a symmetry group $SO(n)$ acting on the equator and all parallel copies of S^{n-1}. Finally, V. Guillemin used an impressive amount of modern techniques (Radon transforms, the Nash-Moser implicit-function theorem, and Fourier-integral operators) to resolve an old problem of Funk; in 1913, Funk showed there were formal series deformations of the canonical metric on S^2 to metrics having all geodesics closed, but the convergence was not proved until 1976 by Guillemin.

There are several other ways in which Besse exploits highly developed structures to study various aspects of manifolds with many closed geodesics. The geometry of conjugate and cut loci is examined to gain insight into Blaschke's conjecture and its generalization to higher dimensions. The conjecture is: a Riemannian manifold for which the geodesics from each point come together again at distance π is a CROSS. For the case of a spherical manifold an affirmative answer was given just in time to be included in Besse's book, through the efforts of M. Berger and J. Kazdan. When such an assumption is made about just one point there are strong topological conclusions derived from Morse theory. The length-spectrum of a Riemannian manifold, that is, the set of lengths of closed geodesics, is closely tied to the spectrum of the Laplacian by use of the techniques of Fourier-integral operators.

The interplay and quantity of good ideas from geometry, topology, analysis, and algebra which are found in Besse's book are astounding. One might guess that it would be impossibly difficult for the bulk of the potential readers, but the style may be justified by the emphasis which our current mathematical training gives to the understanding of structures. Besse has made his book a prototype of how all that structural theory can be specialized to solve problems. In addition, he softens the difficulty by taking pains to summarize the book as a whole and, again, each chapter. The mutual dependence of the chapters is reasonably limited, and the range of background needed varies considerably with the topics, so that there is something valuable and interesting here for almost every mathematician.

Those who wonder why such a knowledgeable author has suddenly appeared with no previous record of publications will find that he is a relative of N. Bourbaki, if they inquire in the right circles.

REFERENCES

1. Ralph Abraham and Jerrold Marsden, *Foundations of mechanics*, 2nd ed., Benjamin Cummings, Reading, Mass., 1978.

2. V. I. Arnol'd, *Mathematical methods of classical mechanics*, Graduate Texts in Math., vol. 60, Springer-Verlag, New York, 1978.

3. N. Bourbaki, *Éléments de mathématique*, Fasc. 24 Livre II, Chapitre 9, *Formes sesquilineares et formes quadratiques*, Hermann, Paris, 1969.

4. S. S. Chern, *From triangles to manifolds*, Amer. Math. Monthly **86** (1979), 339–349.

5. Ian N. Sneddon, *Reivew of* [**2**, **7**], Bull. Amer. Math. Soc. (N.S.) **2** (1980), 346–352.

6. Shlomo Sternberg, *Review of* [**1**], Bull. Amer. Math. Soc. (N.S.) **2** (1980), 378–387.

7. W. Thiring, *Classical dynamical systems*, Springer-Verlag, New York, 1978.

8. A. Weinstein, *On the volume of manifolds all of whose geodesics are closed*, J. Differential Geom. **9** (1974), 513–517.

RICHARD L. BISHOP

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 11, Number 2, October 1984

0273-0979/84 $1.00 + $.25 per page

Percolation theory for mathematicians, by Harry Kesten, Progress in Probability and Statistics, vol. 2, Birkhauser, Boston, Mass., 1982, 423 pp., $30.00. ISBN 3-7643-3107-0

Physicists have enthusiastically embraced percolation models, and a dramatic explosion of physics literature on percolation has occurrred in recent years. This literature is rich in simulations, conjectures, heuristic methods, and a wide variety of applications and variations of the basic models. Mathematicians who experience frustration in tracing the thread of fact through this tangle of conjecture and empirical evidence will appreciate the mathematical rigor in *Percolation theory for mathematicians.*

Percolation models originated in discussions between Broadbent and Hammersley (1957) on the excluded volume problem in polymer chemistry and the design of coal miners' masks. Such topics suggested a probabilistic model for fluid flow in a medium with randomness associated with the medium rather than the fluid. Hence, percolation theory arose as an alternative to the more familiar diffusion models, in which randomness is associated with the fluid.

Models. In a percolation model, the medium is represented by a graph G, which is usually an infinite graph with some regularity of structure. Familiar examples are the square, triangular, and hexagonal lattices in two dimensions, and the cubic lattice in three dimensions. The fluid flow is determined by a random network of vertices and edges in the graph.

The random mechanism may be associated with either the vertices or the edges, so two standard models arise: In the *bond* percolation model, each edge is "occupied" by fluid with probability p and " vacant" with probability $1 - p$,