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CHARACTERISTIC CLASSES OF SURFACE BUNDLES

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In this paper we define characteristic classes of surface bundles, namely smooth fibre bundles whose fibres are a closed orientable surface Σ_g of genus $g \geq 2$, and announce some nontriviality results for them. As a consequence we obtain lower bounds for the Betti numbers of the mapping class group $M(g)$ of Σ_g.

It is known [**EE**] that the connected component of the identity of $\mathrm{Diff}_+ \Sigma_g$, the group of orientation preserving diffeomorphisms of Σ_g, is contractible. Therefore $B\mathrm{Diff}_+ \Sigma_g$ is a $K(M(g), 1)$. Now let ξ be the tangent bundle along the fibres of an *oriented* surface bundle and let $e(\xi)$ be its Euler class. If we apply the Gysin homomorphsm to $e^{i+1}(\xi)$, we obtain an integral cohomology class of the base space of degree $2i$. By naturality this defines certain cohomology classes $e_i \in H^{2i}(M(g)\colon \mathbf{Z})$ $(i = 1, 2, \ldots)$. $M(g)$ acts on $H^1(\Sigma_g; \mathbf{Z})$, preserving the symplectic form given by the cup product, so we obtain a homomorphism $M(g) \to \mathrm{Sp}(2g; \mathbf{Z})$, where $\mathrm{Sp}(2g; \mathbf{Z})$ is the group of all $2g \times 2g$ symplectic matrices with integral entries. This induces a homomorphism $M(g) \to \mathrm{Sp}(2g; \mathbf{R})$. Since $\mathrm{Sp}(2g; \mathbf{R})$ has $\mathrm{U}(g)$ as a maximal compact subgroup, we have a g-dimensional complex vector bundle η on $K(M(g), 1)$. Let $c_i(\eta) \in H^{2i}(M(g); \mathbf{Z})$ be its ith Chern class. From the argument of Atiyah in [**A**] and the fact that η is flat as a real vector bundle, we can conclude

$$e_{2i-1} = (-1)^i (2i/B_i) s_{2i-1}(c(\eta)) \qquad (i = 1, 2, \ldots \text{and coefficients are in } \mathbf{Q}),$$
$$s_{2i}(c(\eta)) = 0$$

where $s_i(c(\eta))$ stands for the characteristic class of η corresponding to the formal sum $\sum_j t_j^i$, and B_i is the ith Bernoulli number. These two relations induce those among monomials of e_{2i-1}'s and the quotient

$$\mathbf{Q}[e_1, e_3, \ldots]/(\text{relations})$$

is naturally isomorphic to the relative Lie algebra cohomology $H^*(\mathfrak{sp}(2g; \mathbf{R}), \mathfrak{u}(g))$, which in turn is *additively* isomorphic to $H^*(S^2 \times S^4 \times \cdots \times S^{2g}; \mathbf{Q})$ (see [**BH**]). It is known that $M(g)$ acts properly discontinuously on the Teichmüller space $T(g) \cong \mathbf{R}^{6g-6}$ with noncompact quotient $\mathcal{M}_g$, the moduli space for Riemann surfaces of genus g. Hence $\mathrm{vcd}(M(g)) \leq 6g - 7$. Thus any monomial of e_i's of degree $\geq 6g - 6$ vanishes. To sum up we have a homomorphism

$$\phi\colon \mathbf{Q}[e_1, e_2, \ldots]/(\text{above relations}) \to H^*(M(g); \mathbf{Q});$$

here we use the letters e_i for both symbolic and actual meanings. Since $\mathrm{vcd}(M(g))$ is conjectured to be $3g - 3$ [**Hv**], ϕ will surely still have a large kernel. Our main results are

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THEOREM 1. *For any $k \in \mathbf{N}$ there exists a natural number $g(k)$ such that the elements $e_1, \ldots, e_k$ are all nontrivial in $H^*(M(g); \mathbf{Q})$ if $g \geq g(k)$.*

COROLLARY 2. *The natural surjective homomorphism $\mathrm{Diff}_+ \Sigma_g \to M(g)$ does not have a right inverse if $g \geq g(3)$ (we can take 86 for $g(3)$). In fact the induced homomorphism $H_{2i}(\mathrm{Diff}_+ \Sigma_g) \to H_{2i}(M(g))$ is not surjective for $i = 3, 4, \ldots, k$ if $g \geq g(k)$ (here we consider $\mathrm{Diff}_+ \Sigma_g$ as a discrete group).*

This result should be compared with the recent affirmative solution of the Nielsen realization problem by Kerckhoff [**Ke**].

THEOREM 3. *For any $k \in \mathbf{N}$ there exists a natural number $g'(k)$ such that the 2ith Betti number $b_{2i}(M(g))$ of $M(g)$, which is equal to $b_{2i}(\mathcal{M}_g)$, is at least i for all $i = 1, \ldots, k$ if $g \geq g'(k)$.*

We mention that Harer [**Ha**] has proved that $b_2(M(g)) = 1$ for $g \geq 5$. Since "half" of our characteristic classes come from $\mathrm{Sp}(2g; \mathbf{Z})$, we also have information on the homomorphism $H_{2i}(M(g)) \to H_{2i}(\mathrm{Sp}(2g; \mathbf{Z}))$. We omit the precise statement.

SKETCH OF PROOFS. The proofs of the above results are given by constructing sufficiently many surface bundles with nontrivial characteristic classes. Roughly speaking we apply the method of Atiyah [**A**] (see also [**Ko**]) iteratively. To be more precise, let $\pi \colon E \to X$ be a surface bundle with fibre Σ_g. We assume that X is an iterated surface bundle. Given $(n, n') \in \mathbf{N} \times \mathbf{N}$, an "$(n, n')$-construction on $\pi \colon E \to X$" is described by the following diagram of surface bundles:

$$
\begin{array}{ccccccccc}
F & \longrightarrow & E_2^* & \longrightarrow & {}'E_1^* & \longrightarrow & E_1^* & \longrightarrow & E^* \\
\downarrow \Sigma'' & & \downarrow \Sigma' & & \downarrow \Sigma' & & \downarrow \Sigma & & \downarrow \Sigma \\
E_2 & \Longrightarrow & E_2 & \longrightarrow & E_1 & \Longrightarrow & E_1 & \longrightarrow & E \\
& & & & & & & & \downarrow \Sigma \\
& & & & & & & & X
\end{array}
$$

Here $E^* \to E$ is the pull back bundle $\pi^*(E)$. E^* contains a cross-section D as the diagonal. $E_1 \to E$ is a covering map which kills first the action of $\pi_1(E)$ on $H^1(\text{fibre}; \mathbf{Z}/nn')$ and then kills $H^1(\ ; \mathbf{Z}/nn')$. $E_1^* \to E^*$ is the pull back by this map. ${}'E_1^* \to E_1^*$ is a fibrewise nn'-fold covering map. $E_2 \to E_1$ is a covering map which satisfies the condition: the homology class of the inverse image D' of D under the map $E_2^* \to E^*$ is divisible by n. The assumption that X is an iterated surface bundle guarantees the existence of such a covering. Finally $F \to E_2^*$ is an n-fold cyclic covering ramified along D'. $F \to E_2$ is a surface bundle with fibre Σ'' whose genus is $n^2 n' g - \frac{1}{2} n(n+1)n' + 1$. The $(2,1)$-construction on the trivial surface bundle $\Sigma_g \to \mathrm{pt}$ is nothing but Atiyah's method in [**A**]. Theorem 1 is proved by calculating e_k of surface bundles which are defined by applying (n_j, n_j')-constructions on $\Sigma_g \to \mathrm{pt}$ successively $(j = 1, \ldots, k)$. It turns out that e_k of such a surface bundle is $(g-1)$ times a nontrivial polynomial of n_j, n_j''s. Since such surface bundles admit multivalued cross-sections, once the statement of Theorem 1 is proved for one g_0, it holds for all $g \geq g_0$. Corollary 2 follows from Theorem 1 and the Bott vanishing theorem [**B**]. We can also compute characteristic classes other than e_k. It turns out that $e_{i_1}^{d_1}\cdots e_{i_s}^{d_s}$ ($\sum_j i_j d_j = k$) is a linear combination of $(g-1), (g-1)^2, \dots, (g-1)^{d_1+\cdots+d_s}$ with coefficients in polynomials of n_j, n'_j's. Theorem 3 follows from this.

It is very likely that these examples of surface bundles are enough to prove the injectivity of ϕ in small degrees. However, necessary computations for that are extremely complicated. Also it seems to be interesting to test the surjectivity of ϕ by examining these examples because it is by no means clear that characteristic numbers of a surface bundle which is obtained by applying an (n, n')-construction on another surface bundle depend only on those of the latter. The details together with these points will appear elsewhere.

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