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REDUCIBILITY OF STANDARD REPRESENTATIONS

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Let G be a real linear reductive group with abelian Cartan subgroups. Unexplained notation, in general, follows [**3** and **6**]. Fix a parabolic subgroup $P = MAN$ of G and a representation δ of M in the limits of the discrete series. The continuous family of representations

$$\pi(\nu) = \mathrm{Ind}_P^G(\delta \otimes \nu \otimes 1) \qquad (\nu \in \hat{A} \cong \mathfrak{a}^*)$$

is a typical series of *standard representations* of G. (These are not, in general, unitary since ν may not be a unitary character of A.) In order to apply certain "continuity arguments" in the study of unitary representations of G, it is necessary to know for which values of ν the representations $\pi(\nu)$ is reducible. We sketch here an explicit answer to this question for classical groups. (Our techniques reduce the problem for exceptional groups to a (long) finite calculation.) The continuity arguments mentioned above require a similar understanding of reducibility for some larger class (it is not yet clear *what* larger class) of induced representations. Some of our techniques also apply to this more general problem.

Write $\overline{\pi}(\nu)$ for the direct sum of the Langlands subquotients of $\pi(\nu)$. These are the irreducible composition factors of $\pi(\nu)$ whose matrix coefficients exhibit the largest possible growth at infinity [**1**]. (Alternatively [**4**], they may be characterized by the fact that their restrictions to a maximal compact subgroup contain representations which are as small as possible.) Obviously $\pi(\nu)$ is reducible if and only if at least one of the following conditions holds: $\overline{\pi}(\nu)$ is reducible; or $\pi(\nu)$ has some composition factor not in $\overline{\pi}(\nu)$. We write the second possiblity as $\pi(\nu) \neq \overline{\pi}(\nu)$. Now Knapp and Zuckerman have determined in [**2**] exactly when the first possibility occurs: ν must belong to one of finitely many linear subspaces in $\mathfrak{a}^*$, which are explicitly described in terms of the inducing representation δ. We must therefore explain when $\pi(\nu) \neq \overline{\pi}(\nu)$.

In writing a Langlands decomposition $P = MAN$, we have implicitly fixed a Cartan involution θ. Choose a θ-stable compact Cartan subgroup $T \subseteq M$ and write $H = TA$ for the corresponding θ-stable Cartan subgroup of G. The representation δ determines (up to conjugacy under $W(M,T)$) a positive root system $\Delta^+(\mathfrak{m}, \mathfrak{t})$ and a Harish-Chandra parameter $\lambda \in \mathfrak{t}^*$. Put

$$\overline{\gamma} = (\lambda, \nu) \in \mathfrak{t}^* + \mathfrak{a}^* \cong \mathfrak{h}^*,$$

$$R(\delta \otimes \nu) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) | \langle \check{\alpha}, \overline{\gamma} \rangle \in \mathbf{Z}\};$$

as usual, $\check{\alpha}$ denotes the coroot $2\alpha/\langle \alpha, \alpha \rangle$.

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The root system $R(\delta \otimes \nu)$ has several additional structures:

(1) θ acts on $R(\delta \otimes \nu)$.

(2) Each θ-fixed (that is, *imaginary*) root is either compact or noncompact.

(3) Each $(-\theta)$-fixed (that is, *real*) root either does or does not satisfy a "parity condition" [**3**].

(4) There is a decomposition

$$R(\delta \otimes \nu) = R^{++}(\delta \otimes \nu) \cup R_0(\delta \otimes \nu) \cup R^{--}(\delta \otimes \nu)$$

of the roots according to whether their inner products with $\overline{\gamma}$ are positive, zero, or negative.

(5) There is a distinguished choice $\Delta^+(\mathfrak{m}, \mathfrak{t})$ of positive imaginary roots.

Fix a nonzero weight $\phi \in \mathfrak{a}^*$ of $\mathfrak{a}$ in $\mathfrak{g}$, and set

$$R(\delta \times \nu)^\phi = \{\alpha \in R(\delta \otimes \nu) | \alpha|_\mathfrak{a} \in \mathbf{R} \cdot \phi\}.$$

This root system inherits all the extra structure of $R(\delta \otimes \nu)$. Choose a positive root system $R_0^+(\delta \otimes \nu)^\phi$ so that

(a) $R_0^+(\delta \otimes \nu)^\phi \supseteq \Delta^+(\mathfrak{m}, \mathfrak{t}) \cap R_0(\delta \otimes \nu)$.

(b) If $\alpha \in R_0(\delta \otimes \nu)^\phi$ and $(-\theta\alpha) \in R^{++}(\delta \otimes \nu)^\phi$, then $\alpha \in R_0^+(\delta \otimes \nu)$.

(c) If α and $-\theta\alpha$ are distinct elements of $R_0(\delta \otimes \nu)^\phi$, then both belong to $R_0^+(\delta \otimes \nu)$, or neither does.

Define

$$^{\mathbf{R}}R^+(\delta \otimes \nu)^\phi = R^{++}(\delta \otimes \nu)^\phi \cup R_0^+(\delta \otimes \nu)^\phi,$$

$$\Pi = {}^{\mathbf{R}}\Pi(\delta \otimes \nu) = \text{simple roots of } {}^{\mathbf{R}}R^+(\delta \otimes \nu)^\phi.$$

If $-\theta$ preserves ${}^{\mathbf{R}}R^+(\delta \otimes \nu)^\phi$, define $C(\delta \otimes \nu)^\phi$ to be the empty root system. Otherwise, we can write

$$\phi = \sum_{\alpha \in \Pi} n_\alpha \alpha,$$

with n_α a nonnegative rational number. We define

$$\Pi_{\text{crit}} = {}^{\mathbf{R}}\Pi(\delta \otimes \nu)^\phi_{\text{crit}} = \{\alpha \in \Pi | n_\alpha \neq 0\},$$

$$C(\delta \otimes \nu)^\phi = \text{span of } \Pi_{\text{crit}},$$

the *critical root system.*

PROPOSITION [**5**]. *There is a connected simple group $\tilde{G}$ with parabolic subgroup $\tilde{P} = \tilde{M}\tilde{A}\tilde{N}$, $\tilde{\delta} \in \hat{\tilde{M}}$, $\tilde{\nu} \in \tilde{\mathfrak{a}}^*$, etc., all unique up to isomorphism, such that*

(a) $\dim \tilde{A} = 1$.

(b) $\tilde{\pi}(\tilde{\nu}) = \operatorname{Ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\delta} \otimes \tilde{\nu} \otimes 1)$ *has integral infinitesimal character.*

(c) $\Delta(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}) = R(\tilde{\delta} \otimes \tilde{\nu}) \cong C(\delta \otimes \nu)^\phi$, *the isomorphism preserving the additional structures* (1)–(5) *described above.*

The critical root system $C(\delta \otimes \nu)^\phi$ is said to be of *reducible type* if the representation $\tilde{\pi}(\tilde{\nu})$ described by the proposition is reducible.

THEOREM. *Let $\pi(\nu) = \operatorname{Ind}_P^G(\delta \otimes \nu)$ be a standard representation as described above. Then $\pi(\nu) \neq \overline{\pi}(\nu)$ ($\pi(\nu)$ is distinct from its Langlands subquotients) if and only if there is a nonzero weight ϕ of $\mathfrak{a}$ in $\mathfrak{g}$ such that the attached critical root system $C(\delta \otimes \nu)^{\phi}$ (defined above) is nonempty and of reducible type.*

This was largely proved (implicitly) in [**3**]. Our solution to the reducibility problem (for classical groups) consists of a list of all critical root systems (together with the additional structures (1)–(5)) which are *not* of reducible type. For complex groups and $\mathrm{GL}(n, \mathbf{R})$, there are no such irreducible critical root systems (except the empty one), so the theorem reduces to results of Zhelobenko [**9**] and Speh [**8**], respectively. Kostant's results [**7**] on reducibility of spherical series may be interpreted as describing certain irreducible critical root systems corresponding to the large region of irreducibility round $\nu = 0$. The group $\tilde{G}$ of the Proposition in these cases has real rank one. This already accounts for many of the irreducible critical root systems. Most of the rest correspond to $\tilde{G}$ of real rank 2.

We also give tables describing the smallest real ν for which $\pi(\nu)$ is reducible when $\dim A = 1$ and use these to study unitarizability of $\overline{\pi}(\nu)$ in that case. Details and proofs will appear elsewhere.

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