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# SIMPLE CLOSED GEODESICS ON $H^+/\Gamma(3)$ ARISE FROM THE MARKOV SPECTRUM

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**1.** Let

$$H^+ = \{z = x + iy \colon y > 0\}$$

be the complex upper half-plane, and let

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod n;\ a, b, c, d \in \mathbf{Z},\ ad - bc = 1 \right\}$$

be the principal congruence subgroup of level $n$ in the modular group $\mathrm{SL}(2, \mathbf{Z}) = \Gamma(1)$. In this note we are concerned with $\Gamma(3)$. Let $S$ be the Riemann surface $H^+/\Gamma(3)$ and let $\pi\colon H^+ \to S$ be the projection map. $S$ is a sphere with four punctures.

A hyperbolic element $\gamma$ is a Möbius transformation of $H^+$ that has two real fixed points; its axis $A_\gamma$ is the circle with center on $\mathbf{R}$ connecting the fixed points. Write $\xi_\gamma, \xi'_\gamma$ for the fixed points of $\gamma$. If $\gamma \in \Gamma(3)$ is hyperbolic, $A_\gamma$ projects to a closed geodesic on $S$; conversely, every closed geodesic on $S$ arises in this way. A *simple* closed geodesic is one that does not intersect itself.

The Markov Spectrum will be described in detail in §2. Here we note the definition of the Markov function $M(\theta)$. For real irrational $\theta$ set

$$M(\theta) = \sup\{c > 0 \colon |\theta - p/q| < 1/cq^2 \text{ for infinitely many reduced fractions } p/q\}. \tag{1.1}$$

In the range $M(\theta) < 3$, $M$ assumes only a denumerably infinite set of values $M_\nu \uparrow 3$. The numbers $M_\nu$ constitute the Markov Spectrum, which we denote by MS.

The connection between simple closed geodesics on $S$ and MS is established in the following way. For $\beta \in \Gamma(3)$ write $A_\gamma \wedge \beta A_\gamma$ to mean $A_\gamma \cap \beta A_\gamma | \neq | \emptyset, A_\gamma$, i.e., the intersection is a single point in $H^+$. The following criterion is easy to prove:

(1.2) $\pi(A_\gamma)$ is nonsimple if and only if $A_\gamma \wedge \beta A_\gamma$ for some $\beta \in \Gamma(3) - \langle\gamma\rangle$.

But in this statement we know nothing about $\beta$ except that if is not elliptic ($\Gamma(3)$ contains no elliptic elements).

THEOREM 1. *If $\pi(A_\gamma)$ is nonsimple, there is a parabolic element $P$ in $\Gamma(3)$ such that $A_\gamma \wedge PA_\gamma$.*

Theorem 1 leads directly to the main result:

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THEOREM 2. *Let $\gamma \in \Gamma(3)$ be hyperbolic. Then $\pi(A_\gamma)$ is simple if and only if $M(\xi_\gamma) = M(\xi'_\gamma) < 3$.*

Since Zagier [**3**] has recently given an asymptotic formula describing this portion of MS, we can deduce from Theorem 2:

COROLLARY 1. *Let $N_S(T)$ be the number of simple closed geodesics on $S$ of hyperbolic length $\leq T$. Then $T \ll N_S(T) \ll T^2$.*

The implied constants are effective. This inequality contrasts with results on $N(T)$, the number of closed geodesics of length $\leq T$, first obtained by H. Huber [**1**].

Returning to Theorem 1, we make use in the proof of the following known facts:

(1.3) A simple loop $L$ contained in $\pi(A_\gamma)$ cannot bound a disk, i.e., each component of $S - L$ must contain at least one puncture.

(1.4) If $L$ bounds a disk with exactly one puncture, then $L$ determines a conjugacy class of parabolic elements of $\Gamma(3)$.

We remark that $\pi(A_\gamma)$ has a finite number of self-intersections, since it is a real-analytic curve.

Now it can be shown that $\pi(A_\gamma)$, assumed nonsimple, contains a simple loop $L$ surrounding a single puncture $p$. We are indebted to A. F. Beardon for a proof of this fact that is shorter and simpler than the original one.

There is a lift of $\pi(A_\gamma)$ lying on $A_\gamma$ and starting from a point $\varsigma$, i.e., the lift is an interval $(\varsigma, \gamma\varsigma)$ of $A_\gamma$. Using (1.4) one can show that there is a parabolic element $P$ in $\Gamma(3)$ such that the lift of $L$ is an interval $(z_0, Pz_0) \subset (\varsigma, \gamma\varsigma)$. Thus $A_\gamma \wedge PA_\gamma$, as asserted.

Full details will follow in a paper written jointly with A. F. Beardon. This paper also contains the following result. Let $\Gamma$ be a finitely generated fuchsian group and let $S = H^+/\Gamma$ be the associated Riemann surface. Then Theorem 1 holds for $S$ if and only if $S$ is of genus zero and has either three or four punctures or deleted disks.

**2.** We now return to the Markov Spectrum (MS); for a fuller account see [**2**, pp. 29–32]. In (1.1) and the following lines we defined MS to be the set of values $\{M_\nu\}$ assumed by the Markov function $M(\theta)$ in the range $M(\theta) < 3$. In order to calculate $M_\nu$ we introduce *Markov triples.* A triple of positive integers $(x, y, z)$ is called a Markov triple if $x^2 + y^2 + z^2 = 3xyz$, $1 \leq x \leq y \leq z$. The first triples are $(1,1,1), (1,1,2), (1,2,5), \ldots$, and the rest can be recursively generated. Order the triples by the size of $z$ so that $1 = z_1 \leq 2 = z_2 \leq \cdots \leq z_\nu \cdots$. With each triple $(x_\nu, y_\nu, z_\nu)$ there is associated a pair of real quadratic conjugates

$$\theta_\nu, \theta'_\nu = \tfrac{1}{2} + y_\nu/x_\nu z_\nu \pm \tfrac{1}{2}(9 - 4/z_\nu^2)^{1/2}, \qquad \nu \geq 1. \tag{2.1}$$

The connection of $M(\theta)$ with $\theta_\nu$ is that

$$M(\theta_\nu) = M_\nu = |\theta_\nu - \theta'_\nu| = (9 - 4/z_\nu^2)^{1/2}. \tag{2.2}$$

We have $M_1 = 5^{1/2}, M_2 = 8^{1/2}, M_3 = (221)^{1/2}/5, \ldots, \to 3$.

Next, introduce the equivalence relation:

$\theta \sim \psi$ if and only if $\psi = (a\theta+b)/(c\theta+d)$ with integers $a, b, c, d$ and $ad - bc = \pm 1$. (2.3)

Then $\theta \sim \psi$ if and only if

(2.4) $$M(\theta) = M(\psi).$$

Moreover, the regular continued fraction expansions of $\theta$ and $\psi$ agree from a certain point on. Also

(2.5) $$M(\theta) < 3 \Rightarrow \theta \sim \theta_\nu \quad \text{for some } \nu \geq 1.$$

Indeed, the definition of MS shows that $M(\theta) = M_\nu = M(\theta_\nu)$, so $\theta \sim \theta_\nu$ by (2.4).

The numbers $\{\theta_\nu\}$, $\{\theta'_\nu\}$, together with their equivalents under (2.3), are called Markov quadratic irrationalities (MQI). Theorem 2 may now be restated.

THEOREM 2′. *$\pi(A_\gamma)$ is simple if and only if $\xi_\gamma$ is equivalent to a MQI.*

We can associate MS to hyperbolic elements of $\Gamma(3)$. For each $\nu$ there is a $\gamma_\nu \in \Gamma(3)$ whose fixed points are $\xi_{\gamma_\nu} = \theta_\nu, \xi'_{\gamma_\nu} = \theta'_\nu$. Namely, dropping the subscript $\nu$, let $\varsigma = 1$ if $z$ is odd, otherwise $\varsigma = 1/2$. Define

(2.6) $$B = \begin{pmatrix} (N + x(2y+xz)\varsigma M)2^{-1} & (2x^2z - 4xy + z)\varsigma M \\ x^2z\varsigma M & (N - x(2y+xz)\varsigma M)2^{-1} \end{pmatrix},$$

where $M > 0$ is the smallest integral solution of the Pell equation

$$x^4(9z^2-4)\varsigma^2M^2 + 4 = N^2.$$

Then it can be shown that $B$ is the $\Gamma(1)$-primitive matrix fixing $\xi, \xi'$. Moreover, $B \in \Gamma(3)$ if $3|M$, otherwise $B^2 \in \Gamma(3)$. But the first case never occurs, so $B^2$ is the $\Gamma(3)$-primitive matrix fixing $\xi, \xi'$.

By abuse of notation we say $\gamma \in$ MS if $\gamma \in \Gamma(3)$ and $\xi_\gamma \sim \theta_\nu$ for some $\nu \geq 1$. If $\gamma \in$ MS so does $V\gamma V^{-1}, V \in \Gamma(1)$, since $V\gamma V^{-1} \in \Gamma(3)$ by normality of $\Gamma(3)$ in $\Gamma(1)$ and $\xi_{V\gamma V^{-1}} = V\xi_\gamma \sim V\theta_\nu \sim \theta_\nu$. That is,

(2.7) the conjugacy class of $\gamma$ in $\Gamma(1)$ belongs to MS if $\gamma \in$ MS.

We now prove Theorem 2. Suppose $\pi(A_\gamma)$ is nonsimple; then by Theorem 1 there is a $\delta$ conjugate to $\gamma$ in $\Gamma(3)$ for which $A_\delta \wedge S^3 A_\delta$, i.e., $|\xi_\delta - \xi_{\delta'}| > 3$. By a translation in $\Gamma(1)$ we may assume $-1 < \xi'_\delta < 0$; then $\xi_\delta > \xi'_\delta + 3 > 1$. Thus $\xi_\delta$ is "reduced" [**4**, p. 73] and the regular continued fraction of $\xi_\delta$ is pure periodic; also $\xi'_\delta$. Let $\xi_\delta = \overline{(b_0, b_1, \ldots, b_{k-1})}$ for $k \geq 1$; then $-1/\xi'_\delta = \overline{(b_{k-1}, \ldots, b_0)}$ [**4**, p. 76]. Here $b_{nk+\nu} = b_\nu$ for $0 \leq \nu < k$, $n \geq 0$. Set

$$m_\mu = \overline{(b_\mu, b_{\mu+1}, \ldots, b_{\mu+k-1})} + (0, \overline{b_{\mu-1}, b_{\mu-2}, \ldots, b_{\mu-k}}), \qquad \mu \geq k.$$

By periodicity $m_\mu = m_{\mu+k}$. Moreover, $M(\xi_\delta) = \overline{\lim}_{\mu\to\infty} m_\mu$ [**2**, p. 29].

Therefore, for all $\varepsilon > 0$ and $n > N$,

$$3 < \xi_\delta - \xi'_\delta = m_k = m_{nk} < \overline{\lim_{\mu\to\infty}}\, m_\mu + \varepsilon < M(\xi_\delta) + \varepsilon,$$

implying

$$M(\xi_\gamma) = M(\xi_\delta) > 3,$$

as asserted.

Conversely, assume $\pi(A_\gamma)$ is simple. Then certainly $|\xi_\gamma - \xi'_\gamma| \leq 3$, otherwise $A_\gamma \wedge S^3 A_\gamma$. Since $\pi(A_\gamma)$ is simple if and only if $\pi(VA_\gamma) = \pi(A_{V\gamma V^{-1}})$ is simple for all $V \in \Gamma(1)$—because $\Gamma(3) \lhd \Gamma(1)$—we have

$$(*) \qquad |V\xi_\gamma - V\xi'_\gamma| \leq 3, \qquad V \in \Gamma(1).$$

Assuming $\gamma \notin$ MS we shall produce a $V \in \Gamma(1)$ that contradicts $(*)$.

At this point we observe that $M(\xi_\gamma) \neq 3$ for any $\gamma \in \Gamma(1)$. Indeed, $\xi_\gamma$ is a quadratic irrationality and $M(\theta)$ is never 3 if $\theta$ is a quadratic irrational [**2**, p. 32]. It follows that $\gamma \notin$ MS implies $M(\xi_\gamma) > 3$, that is,

$$|\xi_\gamma - p_n/q_n| < 1/(3+h)q_n^2, \qquad (p_n, q_n) = 1,$$

for some $h > 0$, on a sequence $q_n \to \infty$. Write $V_n = (q'_n, -p'_n : q_n, -p_n) \in \Gamma(1)$. Then with $\xi_\gamma = \xi$, $\xi'_\gamma = \xi'$,

$$\begin{aligned} |V_n\xi - V_n\xi'| &= \frac{|\xi - \xi'|}{q_n^2|\xi - p_n/q_n|\,|\xi' - p_n/q_n|} > \frac{(3+h)|\xi - \xi'|}{|\xi' - p_n/q_n|} \\ &\geq \frac{(3+h)|\xi - \xi'|}{|\xi' - \xi| + |\xi - p_n/q_n|} > \frac{3+h}{1 + 1/3q_n^2|\xi - \xi'|} > 3, \end{aligned}$$

for $n \geq n_0$. For $V = V_{n_0}$ we have a contradiction to $(*)$.

We close with a comment on Corollary 1. The existence of long simple geodesics on $H^+/\Gamma(3)$ is not hard to prove topologically. The feature of Corollary 1 is that the lengths are known explicitly: they are

$$\text{length}\, A_{B_\nu^2} = 2\log\frac{t_\nu + \sqrt{t_\nu^2 - 4}}{2}, \qquad t_\nu = \text{trace}\, B_\nu^2.$$

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