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WEIGHTED POLYNOMIALS ON FINITE AND INFINITE INTERVALS: A UNIFIED APPROACH

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1. Introduction. As described in the survey article [**6**], the study of "incomplete polynomials", as introduced by G. G. Lorentz [**4**] in 1976, leads to results on the asymptotic properties of polynomials orthogonal on an infinite interval (cf. [**5**]) and to theorems on the convergence of "ray sequences" of Padé approximants for Stieltjes functions. Here we present a generalization of the theory for incomplete polynomials which unifies many of the previous results. The essential question which serves as the starting point for the investigation is the following:[1]

Suppose $w(x)$ is a nonnegative weight function continuous on its support $\Sigma \subset \mathbf{R} = (-\infty, \infty)$. (By the *support* of w we mean the *closure* of the set where w is positive.) Assume that $w(x)$ vanishes at points of Σ; that is, $Z := \{x \in \Sigma : w(x) = 0\} \neq \emptyset$ (or, in case Σ is unbounded, then $|x|w(x) \to 0$ as $|x| \to \infty$). If P_n is an arbitrary polynomial of degree at most n, then the sup norm over Σ of the weighted polynomial $[w(x)]^n P_n(x)$ actually "lives" on some compact set $S \subset \Sigma - Z$ which is independent of n and P_n. The question is to determine the smallest such set S.

For example, if $w(x) = x^{\theta/(1-\theta)}$ with $\Sigma = [0, 1], 0 < \theta < 1$, then, as shown in [**2, 8**], S is the subinterval $[\theta^2, 1]$.

In this paper we use potential theoretic methods to show how S can be obtained for a class of weight functions. The assumptions on w are given in

DEFINITION 1.1. Let $w\colon \mathbf{R} \to [0, +\infty)$. We say that w is an *admissible weight function* if each of the following properties holds:

(i) $\Sigma := \operatorname{supp}(w)$ has positive capacity.

(ii) The restriction of w to Σ is continuous on Σ.

(iii) The set $Z := \{x \in \Sigma\colon w(x) = 0\}$ has capacity zero.

(iv) If Σ is unbounded, then $|x|w(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$.

Here, and throughout the paper, the term "capacity" means inner logarithmic capacity (cf. [**10**, p. 55]). For any set $E \subset \mathbf{R}^2$, its capacity will be denoted by $C(E)$. If K is a compact set with positive capacity, then ν_K denotes the unique unit equilibrium measure on K with the property that (cf. [**10**, p. 60])

$$\int_K \log|x - t|\, d\nu_K(t) = \log C(K) \tag{1.1}$$

quasi-everywhere (q.e.) on K. (A property is said to hold q.e. on a set A if the subset E of A where it does not hold satisfies $C(E) = 0$.)

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For an admissible weight w, we always set

$$Q(x) := \log(1/w(x)). \tag{1.2}$$

Finally, if $K \subset \Sigma - Z$ is compact and $C(K) > 0$, we define the *F-functional* of K by the formula

$$F(K) := \log C(K) - \int_K Q\,d\nu_K. \tag{1.3}$$

The theorems of §2 show that, for a class of weight functions, S is derived by maximizing the F-functional. Also, if π_m denotes the collection of all polynomials of degree at most m and $\|\cdot\|_A$ denotes the sup norm over a set A, we describe the asymptotic behavior of the errors in the weighted Chebyshev problem

$$E_n(w) := \inf\{\|[w(x)]^n\{x^n - p_{n-1}(x)\}\|_\Sigma : p_{n-1} \in \pi_{n-1}\}, \quad n = 1, 2, \ldots, \tag{1.4}$$

as well as asymptotic properties (as $n \to \infty$) of the extremal polynomials $T_n(x; w) = x^n + \cdots \in \pi_n$ which satisfy

$$E_n(w) = \|[w(x)]^n T_n(x; w)\|_\Sigma, \qquad n = 1, 2, \ldots. \tag{1.5}$$

2. Statements of main results.

THEOREM 2.1. *Let w be an admissible weight function with support Σ. Then there exists a compact set $S \subset \Sigma - Z$ with $C(S) > 0$ that has the following properties.*

(a) *For every compact set $K \subset \Sigma - Z$ with $C(K) > 0$,*

$$F(K) \le F(S), \tag{2.1}$$

where F is defined in (1.3).

(b) *If equality holds in* (2.1), *then $S \subset K$.*

(c) *For any positive integer n, if $P_n \in \pi_n$ and the inequality*

$$|[w(x)]^n P_n(x)| \le M \qquad (M = \text{constant}) \tag{2.2}$$

holds q.e. on S, then it holds q.e. on Σ.

(d) *The errors $E_n(w)$ defined in* (1.4) *satisfy*

$$[E_n(w)]^{1/n} \ge \exp(F(S)), \quad \forall n = 1, 2, \ldots. \tag{2.3}$$

Clearly properties (a) and (b) uniquely determine the set $S = S(w)$ of Theorem 2.1. In the special case when $w(x) \equiv 1$ on Σ and Σ is compact, then S is just the support of the equilibrium measure ν_Σ for Σ.

Of practical importance is the characterization of S given in

THEOREM 2.2. *Assume that, in Theorem* 2.1, *the set $\Sigma - Z$ is the finite union of disjoint nondegenerate intervals and that $Q(x)$ of* (1.2) *is convex in each of the components of $\Sigma - Z$. Then the following additional properties hold.*

(a) *The compact set S of Theorem* 2.1 *is the finite union of nondegenerate disjoint closed intervals, at most one in each component of $\Sigma - Z$.*

(b) *Equality holds in* (2.1) *if and only if* $\mathcal{S} \subset K$ *and* $C(K - \mathcal{S}) = 0$.

(c) *For any positive integer* n, *if* $P_n \in \pi_n$, *then*

$$\|[w(x)]^n P_n(x)\|_\Sigma = \|[w(x)]^n P_n(x)\|_{\mathcal{S}}. \tag{2.4}$$

(d) *The errors* $E_n(w)$ *of* (1.4) *satisfy*

$$\lim_{n\to\infty} [E_n(w)]^{1/n} = \exp(F(\mathcal{S})). \tag{2.5}$$

The proof of Theorem 2.1 follows by showing that $\mathcal{S}$ is actually the support of a measure which solves an extremal problem for generalized energy integrals, as we now describe. Let $\mathcal{M}(\Sigma)$ denote the collection of all positive unit Borel measures μ with $\operatorname{supp}(\mu) \subset \Sigma$, and define

$$I_w[\mu] := \int\int [\log|x-t| - Q(x) - Q(t)]\, d\mu(x)\, d\mu(t) \tag{2.6}$$

for $\mu \in \mathcal{M}(\Sigma)$. Following methods of Frostman (cf. [**10**]) we obtain

THEOREM 2.3. *Let* w *be an admissible weight function with support* Σ *and let*

$$V_w := \sup\{I_w[\mu] : \mu \in \mathcal{M}(\Sigma)\}. \tag{2.7}$$

Then there exists a unique measure $\mu_w \in \mathcal{M}(\Sigma)$ *such that* $I_w[\mu_w] = V_w$. *Moreover,* $S_w := \operatorname{supp}(\mu_w)$ *satisfies all the properties stated in Theorem* 2.1; *that is,* $S_w = \mathcal{S}$.

Concerning the limiting distribution of the zeros of the extremal polynomials $T_n(x; w)$ we have

THEOREM 2.4. *With the assumptions of Theorem* 2.2, *let* $\{x_{k,n}\}_{k=1}^n$ *denote the zeros of the extremal polynomial* $T_n(x; w)$ *of* (1.5), *and let* ν_n *be the associated unit Borel measure defined by*

$$\nu_n(\mathcal{B}) := (1/n)|\{k : x_{k,n} \in \mathcal{B}\}|.$$

Then, in the weak star topology,

$$\lim_{n\to\infty} \nu_n = \mu_w, \tag{2.8}$$

where μ_w *is the extremal measure of Theorem* 2.3. *Furthermore,*

$$\lim_{n\to\infty} |T_n(z; w)|^{1/n} = \exp\left(\int \log|z-t|\, d\mu_w(t)\right) \tag{2.9}$$

uniformly on every compact set of the plane disjoint from the convex hull $[\lambda, \tau]$ *of* $\mathcal{S}$.

3. Applications. For Jacobi weights of the form $w(x) = x^{\theta/(1-\theta)}, 0 < \theta < 1, \Sigma = [0, 1]$, or $w(x) = (1-x)^{\lambda_1}(1+x)^{\lambda_2}, \lambda_1, \lambda_2 > 0, \Sigma = [-1, 1]$, maximizing the associated F-functional leads to the results of [**2, 9, 3** and **7**] concerning incomplete polynomials.

For a weight W on $\mathbf{R}$ of the form $W(x) = \exp(-q(x))$, where $q(x)$ is even and convex on $\mathbf{R}$ and $q(x)/\ln x \to \infty$ as $x \to \infty$, we can also analyze the extremal problems

$$e_n(W) := \inf\{\|W(x)\{x^n - p_{n-1}(x)\}\|_{\mathbf{R}} : p_{n-1} \in \pi_{n-1}\}, \qquad n = 1, 2, \ldots, \tag{3.1}$$

and the corresponding extremal polynomials $t_n(x;W) = x^n + \cdots \in \pi_n$ satisfying $e_n(W) = \|W(x)t_n(x;W)\|_{\mathbf{R}}$. After maximizing the appropriate F-functional, Theorem 2.2(c) yields

$$\|W(x)P_n(x)\|_{\mathbf{R}} = \|W(x)P_n(x)\|_{[-a_n,a_n]}, \quad \forall P_n \in \pi_n, \tag{3.2}$$

where $a = a_n$ is a root of the equation

$$n = \frac{2}{\pi}\int_0^1 \frac{axq'(ax)}{\sqrt{1-x^2}}\,dx. \tag{3.3}$$

Letting $w_n(x) := \exp(-q(a_n x)/n)$ with $\Sigma_n := \operatorname{supp}(w_n) = [-1,\ 1]$, it follows from (3.2) that

$$e_n(W) = a_n^n E_n(w_n), \qquad t_n(a_n x; W) = a_n^n T_n(x; w_n).$$

If the weights w_n converge uniformly to an admissible weight w on $[-1,\ 1]$, it can be shown that the asymptotic behaviors (as $n \to \infty$) of $E_n(w_n)$ and $T_n(x;w_n)$ are the same as that for $E_n(w)$ and $T_n(x;w)$. These facts lead to the results of [**5**] for $W(x) = \exp(-|x|^\alpha), \alpha \geq 1$, as well as to L^∞-analogue of the L^2-results in [**1**] for $W(x) = \exp(-\exp|x|)$.

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