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Theory of charges, a study of finitely additive measures, by K. P. S. Bhaskara Rao and M. Bhaskara Rao, Academic Press, London, 1983, x + 315 pp., $55.00. ISBN 0-1209-5780-9

A charge is a finitely additive, extended real-valued set function defined on a field of sets. The notion is thus a familiar one even to those who may not have used the term. But why should we study finitely additive measures? Haven't Borel and Lebesgue made them obsolete? We have become so accustomed to countable additivity that most of us take it for granted and feel we would be lost without it. Nevertheless, no less an authority than S. Bochner is quoted as having remarked that finitely additive measures are more interesting, more difficult to handle, and perhaps more important than countably additive ones.

Everyone knows that density is a natural measure in the set of positive integers, and that it has proved very useful in number theory despite the fact that it is only finitely additive. Sometimes density is linked to a countably additive measure. For example, under an ergodic transformation of a normalized measure space, almost all points generate sequences of images that fall in any given measurable set with a frequency (that is, density) equal to the measure of the set. The law of large numbers establishes a similar link between a countably additive probability and densities on almost all sample sequences.

If countable additivity were really indispensable one might wonder how mathematicians managed to get along without it for so long. Of course, length, area and volume are actually countably additive, although this fact was not fully appreciated or exploited until the end of the last century. There are other circumstances in which countable additivity comes as a bonus; for example, when the domain is the field of closed open subsets of a compact space. As a consequence, any charge can be represented by a countably additive charge on a corresponding Stone space, but this representation is too esoteric to be of much use except for special purposes.

It is a remarkable fact that countable additivity is sometimes forced by an invariance requirement. D. Sullivan and G. A. Margulis have recently shown that, for $n \geqslant 3$, Lebesgue measure is the only finitely additive measure on the bounded measurable subsets of R^n that normalizes the unit cube and is isometry-invariant, thus settling a very old and classical problem of Ruziewicz.

The situation is different in R^2, where Banach showed there are many such measures. But even in this case uniqueness can be restored by requiring invariance under the larger group of unimodular affine transformations; in fact, S. Wagon [3] has shown that it is sufficient to require invariance under the single shear transformation $(x, y) \mapsto (x + y, y)$ in addition to isometry-invariance. These results are too recent to have been included in the book under review, and they would fall somewhat outside its scope anyway. The book deals primarily with the general theory of charges, their classification, extension, and relation with other functionals, not with their invariance or ergodic properties. For instance, Banach's invariant extensions of Lebesgue measure to all subsets of R^1 and R^2 are not discussed.

It may come as a surprise to many readers to learn how much of the standard theory of measure and integration can be carried through assuming only finite additivity. Integration can be defined as usual for simple functions and then extended to limits of sequences of simple functions that converge in measure (here called hazy convergence) and also in the L_1-pseudometric. This generalized notion of integral is not new; it may be found, for example, in Dunford-Schwartz [1, p. 112], but its theory is developed here in considerably greater detail. Some results (for example, Jordan decomposition) carry over without change. Others (Hahn decomposition) need to be reformulated in an ε-approximation form. Still others (Vitali-Hahn-Saks theorem) require only that the domain be a σ-field. Finer distinctions need to be drawn. For example, two ordinarily equivalent notions of absolute continuity are no longer equivalent, even for bounded charges: namely, $\nu(E) = 0$ when $|\mu|(E) = 0$, and $|\nu(E)| < \varepsilon$ when $|\mu|(E) <$ some δ. The latter turns out to be the more useful, and is denoted by $\nu \ll \mu$. The Radon-Nikodym theorem takes the following form, due to Bochner: If ν and μ are bounded charges on a field $\mathscr{F}$, and $\nu \ll \mu$, then for each $\varepsilon > 0$ there exists a simple function f such that $|\nu(F) - \int_F f\, d\mu| < \varepsilon$ for all $F \in \mathscr{F}$. The Vitali-Hahn-Saks theorem reads: If μ_n is a sequence of bounded charges on a σ-field $\mathscr{F}$ and $\mu_n(A)$ converges to a real number $\mu(A)$ for each $A \in \mathscr{F}$, then μ is a bounded charge on $\mathscr{F}$, $\{\mu_n\}$ is uniformly additive, and if ν is any bounded charge on $\mathscr{F}$ such that $\mu_n \ll \nu$ for each n, then this relation holds uniformly with respect to n.

The functional analysis of L_p-spaces and their completions is studied, and bounded charges on a Boolean algebra $\mathscr{A}$ receive some attention. For example, the quotient algebra modulo null elements is complete provided $\mathscr{A}$ has the Seever property, a property which is weaker than σ-completeness but sufficient to establish a far-reaching generalization of the uniform boundedness theorem.

The authors have organized a large body of material which is widely scattered in the literature and deserves to be better known. An added section of notes and comments and a carefully annotated bibliography provide the reader with valuable guidance and indications. The exposition is uniformly clear and there is an abundance of examples. It is not surprising that some of these are nonconstructive. The connection between 0-1 valued charges and ultrafilters is well known, and the fact that such a charge can be defined for all subsets of a set and still be equal to zero for every finite subset is useful to remember.

An interesting question, to which the authors have made substantial contributions, concerns what can be said about the range of a charge. If the range is bounded and infinite, then it is dense in itself; if, in addition, the domain is a σ-field, the range contains a dense sequence of perfect sets. However, the range need not be a Borel set; by applying Kolmogorov's zero-one law and its category analogue, the authors show that on any infinite σ-field there exists a probability charge whose range is not Lebesgue measurable and does not have the property of Baire. Here is one example. Let $\mathscr{F}$ be the field of all subsets of $N = \{1, 2, \ldots\}$. For each $A \in \mathscr{F}$ define $\mu_1(A) = \Sigma\{2^{-n}: n \in A\}$, and let μ_0 be a 0-1 valued charge on $\mathscr{F}$ that is equal to zero for every finite set in $\mathscr{F}$. Then $\mu = \frac{1}{2}(\mu_0 + \mu_1)$ is a probability charge on $\mathscr{F}$ whose range has the stated properties. In fact, that part of the range of 2μ that is contained in $(1, 2]$ coincides with a set whose nonmeasurability was proved by Sierpiński [2].

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A problem seminar, by Donald J. Newman, Problem Books in Mathematics, Springer-Verlag, New York, 1982, 113 pp., $12.95. ISBN 0-3879-0765-3

> There was once a bumper sticker that read, "Remember the good old days when *air* was clean and *sex* was dirty?" Indeed, some of us are old enough to remember not only *those* good old days, but even the days when Math was *fun*(!), not the ponderous THEOREM. PROOF. THEOREM. PROOF,..., but the whimsical, "I've got a good problem."
>
> Why did the mood change? What misguided educational philosophy transformed graduate mathematics from a passionate activity to a form of passive scholarship?
>
> In less sentimental terms, why have the graduate schools dropped the Problem Seminar? We therefore offer "A Problem Seminar" to those students who haven't enjoyed the fun and games of problem solving. (Preface to *A problem seminar*).

1. Opening shots. *A problem seminar* is, pound for pound, the finest collection of the problem-solver's art that I have ever read. It is a master class conducted by a man completely in command of his methods. Unfortunately, it is severely compromised by several relatively superficial failings. These failings