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THE GEOMETRY OF THE MODULI SPACE OF RIEMANN SURFACES

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We wish to describe how the hyperbolic geometry of a Riemann surface of genus g, $g \geq 2$, leads to a symplectic geometry on T_g, the genus g Teichmüller space, and $\overline{\mathcal{M}}_g$, the moduli space of genus g stable curves. The symplectic structure has three elements: the Weil-Petersson Kähler form, the Fenchel-Nielsen vector fields t_*, and the geodesic length functions l_*. Weil introduced a Kähler metric for T_g based on the Petersson product for automorphic forms; the metric is invariant under the covering of T_g onto $\mathcal{M}_g$, the classical moduli space of Riemann surfaces [**1**].

To a geodesic α on a marked Riemann surface $\hat{R}$ with hyperbolic metric is associated the Fenchel-Nielsen vector field t_α on T_g; t_α is the infinitesimal generator of the flow given by the Fenchel-Nielsen (fractional twist) deformation for α [**7**]. The infinitesimal generators of Thurston's earthquake flows form the completion (in the compact-open topology) of the Fenchel-Nielsen vector fields [**6**]. A basic invariant of the geodesic α on a marked surface $\hat{R}$ is its length $l_\alpha(\hat{R})$; the exterior derivative dl_α is a 1-form on T_g. We have the following formulas for the Weil-Petersson Kähler form ω [**8**].

THEOREM 1.

$$\text{(i)} \qquad \omega(t_\alpha, \ ) = -dl_\alpha,$$

$$\text{(ii)} \qquad \omega(t_\alpha, t_\beta) = \sum_{p \in \alpha \# \beta} \cos\theta_p,$$

$$\text{(iii)} \qquad t_\alpha t_\beta l_\gamma = \sum_{(p,q)\in \alpha\#\gamma \times \beta\#\gamma} \frac{e^{l_1} + e^{l_2}}{2(e^{l_\gamma} - 1)} \sin\theta_p \sin\theta_q - \sum_{(r,s)\in \alpha\#\beta \times \beta\#\gamma} \frac{e^{m_1} + e^{m_2}}{2(e^{l_\beta} - 1)} \sin\theta_r \sin\theta_s.$$

An immediate consequence of (i) is that the symplectic form ω is invariant under the flow of a Fenchel-Nielsen vector field. The r.h.s. of (ii), evaluated at $\hat{R} \in T_g$, is the sum of the cosines of the angles at the intersections of the geodesics α and β on the surface $\hat{R}$. Similarly the r.h.s. of (iii) is a sum of trigonometric invariants for pairs of intersections; l_1 and l_2 are the lengths of the segments on γ defined by p, q and, likewise, for m_1 and m_2 relative to β. Finally (i) and (ii) may be combined to show that the renormalized vector

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fields, $N_\alpha = (4\sinh l_\alpha/2)t_\alpha$, span a Lie algebra over the integers, which admits a purely topological characterization [**8**].

Fenchel-Nielsen introduced real analytic coordinates for T_g as the parameters for the construction of a hyperbolic metric on a genus g surface. The twist-length coordinates (τ_j, l_j) vary freely: $\tau_j \in \mathbf{R}$, $l_j \in \mathbf{R}^+$ [**6, 10**]. An immediate consequence is that T_g is a cell: $T_g \approx (\mathbf{R}^+ \times \mathbf{R})^{3g-3}$. The Kähler form ω is given simply in Fenchel-Nielsen coordinates [**10**].

THEOREM 2. $\omega = -\sum_j d\tau_j \wedge dl_j$.

In particular, the data (ω, τ_j, l_j) represent a completely integrable Hamiltonian system. Using the above formula it is readily checked that ω extends to a symplectic form on $\overline{\mathcal{M}}_g$, the moduli space of stable curves (hyperbolic Riemann surfaces with nodes). Denote by ω^{FN} this extension of ω to $\overline{\mathcal{M}}_g$.

Our most recent efforts have been directed towards establishing the following.

THEOREM 3. *An integral multiple of* ω/π^2 *is the Chern form of a positive line bundle over* $\overline{\mathcal{M}}_g$.

Let us sketch the development of this result. In [**9**], using formula (i) above, the integral $\int_{\mathcal{M}_{1,1}} \omega$, $\mathcal{M}_{1,1}$ the moduli space of once punctured tori, was calculated: the value is $\pi^2/6$. Masur showed that the Weil-Petersson metric in complex coordinates is actually singular for vectors transverse to $\mathcal{D} = \overline{\mathcal{M}}_g - \mathcal{M}_g$, the divisor of surfaces with nodes, [**5**]. Nevertheless in [**10**] it was shown that $\omega^{\mathbf{C}}$, the extension to $\overline{\mathcal{M}}_g$ of the Kähler form considered in complex coordinates, is a closed, type $(1,1)$ current and $\omega^{\mathbf{C}}$ and ω^{FN} represent the same cohomology class in $H^2(\overline{\mathcal{M}}_g; \mathbf{R})$. In [**11**] we considered the generalization of the earlier $\pi^2/6$ result: $\omega/\pi^2 \in H^2(\overline{\mathcal{M}}_g; \mathbf{Q})$. But first note that the divisor $\mathcal{D} = \overline{\mathcal{M}}_g - \mathcal{M}_g$ is reducible, $\mathcal{D} = \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{\lfloor g/2 \rfloor}$, where the generic surface R represented in $\mathcal{D}_k$ has one node separating R into components of genus k and genus $g-k$. Certainly the divisors $\mathcal{D}_k$ define cohomology classes in $H_{6g-8}(\overline{\mathcal{M}}_g)$, and by Poincaré duality ω also determines a class in $H_{6g-8}(\overline{\mathcal{M}}_g)$.

THEOREM 4. $\{\omega/\pi^2, \mathcal{D}_0, \ldots, \mathcal{D}_{\lfloor g/2 \rfloor}\}$ *is a basis for* $H_{6g-8}(\overline{\mathcal{M}}_g; \mathbf{Q})$.

The plan of the proof of Theorem 4 is straightforward. By the work of Harer [**4**] and an application of Mayer-Vietoris, one checks beforehand that $H_2(\overline{\mathcal{M}}_g; \mathbf{Q})$ has rank $2 + \lfloor g/2 \rfloor$. A candidate basis is then presented for each of $H_2(\overline{\mathcal{M}}_g)$ and $H_{6g-8}(\overline{\mathcal{M}}_g)$, and the intersection pairing is evaluated [**11**]. The pairing is nonsingular and so bases have been given for the homology groups H_2 and H_{6g-8}. By similar techniques it may also be shown that the rank of $H_{2k}(\overline{\mathcal{M}}_g)$, $k < g$, is at least

$$\frac{1}{2}\binom{g-1}{k}.$$

To complete the proof of Theorem 3 it only remains to check that ω is cohomologous in the sense of currents to a closed, positive, $(1,1)$ form. Let Ω be that multiple of ω/π^2 which represents a class in $H^2(\overline{\mathcal{M}}_g; \mathbf{Z})$. Now the formalism of Chern classes extends to currents: Ω determines the class of a line bundle over $\overline{\mathcal{M}}_g$. By direct potential theoretic estimation it is shown that Ω is the curvature form, again in the sense of currents, of a *continuous* metric in a line bundle [**12**]. Next it is checked that ω is bounded below by a smooth positive $(1,1)$ form; Ω is a positive current. Finally, standard techniques from the study of plurisubharmonic functions may be used to complete the argument [**12**].

$\overline{\mathcal{M}}_g$ is a compact, complex V-manifold and so we may refer to Baily's version of the Kodaira Imbedding Theorem [**2**].

THEOREM 5. *The positive line bundle associated to the Weil-Petersson Kähler form gives rise to a projective embedding:* $\overline{\mathcal{M}}_g \hookrightarrow \mathbf{C}P^n$.

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