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JULIA SETS AND BIFURCATION DIAGRAMS FOR EXPONENTIAL MAPS

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ABSTRACT. We describe some of the bifurcations that occur in the family of entire maps $E_\lambda(z) = \lambda \exp(z)$. When $\lambda = 1$, it is known that $J(E_\lambda) = \mathbf{C}$. We show that there are many other values for which this happens. However, in each case, there are nearby λ-values for which $J(E_\lambda)$ is nowhere dense.

Let $F(z)$ be an entire transcendental function. The Julia set of F, denoted by $J(F)$, is the set of points at which the family of iterates of F (i.e. F, $F \circ F = F^2$, $F^3, \ldots$) fails to be a normal family. Equivalently, $J(F)$ is the closure of the set of nonattracting periodic points of F. It is known **[F]** that $J(F)$ is a closed, nonempty, perfect set which is invariant under F and all branches of F^{-1}. Moreover, most of the interesting chaotic dynamics of the map occur on the Julia set.

There has been much recent work on the structure of the Julia set of complex analytic functions **[B, DH, R, S, MSS]**. Most of this work is restricted to the polynomial or rational function case. Our goal in this note is to point out that, while entire functions share many of the properties of these maps, there are several significant differences.

We will concentrate on the one-parameter family of maps $E_\lambda(z) = \lambda \exp(z)$ where $\lambda \in \mathbf{C}$. Similar results hold for other classes of entire maps, e.g. $z \to a + b\sin(z)$ and $z \to Q(z)\exp(P(z))$ where P and Q are polynomials. A major difference between the exponential family and polynomials is the possibility that $J(E_\lambda) = \mathbf{C}$. Indeed, Misiurewicz **[M]** has recently shown that $J(e^z) = \mathbf{C}$, answering affirmatively a sixty year old question of Fatou **[F]**. It turns out that this is a common occurrence for entire maps.

1. The exponential family. The family of functions E_λ is a natural family of complex analytic maps in the sense that any entire function which is topologically conjugate to some E_λ is in fact affinely equivalent to a member of this family. This essentially follows from the fact that E_λ has no critical points and one omitted value. The orbit of the omitted value at 0 is the crucial factor which governs the dynamics of E_λ.

PROPOSITION. 1. *If* $E_\lambda^n(0) \to \infty$, *then* $J(E_\lambda) = \mathbf{C}$.

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2. *If* 0 *is preperiodic (i.e.* $E_\lambda^n(0)$ *is periodic for some* $n \geq 1$*), then again* $J(E_\lambda) = \mathbf{C}$.

3. *If* E_λ *has an attracting periodic orbit, then* 0 *lies in the basin of attraction of this orbit.*

The proofs of 1 and 2 rely on an extension of the classification theorem of Sullivan [S] to the case of E_λ. Part 3 is classical and implies that any E_λ can have at most one attracting periodic orbit.

COROLLARY. 1. $J(E_\lambda) = \mathbf{C}$ *if* $\lambda > 1/e$.
2. $J(E_\lambda) = \mathbf{C}$ *if* $\lambda = k\pi i$, $k \in \mathbf{Z}$.

PROOF. One checks easily that $E_\lambda^n(0) \to \infty$ if $\lambda > 1/e$. If $\lambda = 2k\pi i$, then $E_\lambda^2(0) = 2k\pi i = E_\lambda(0)$. If $\lambda = (2k+1)\pi i$, then $E_\lambda^2(0) = -(2k+1)\pi i = E_\lambda^3(0)$. Hence the proposition applies.

COROLLARY. *There is a Cantor set of curves in the* λ*-plane for which* $J(E_\lambda) = \mathbf{C}$.

The proof of this corollary relies heavily on the structure of the Julia set for E_λ. It is known [**DK**] that repelling periodic points lie at the ends of curves or "hairs" which stretch to infinity. All points on these hairs except the periodic point tend to infinity under iteration of E_λ. One then arranges for $\lambda = E_\lambda(0)$ to lie on such a curve so that $E_\lambda^n(0) \to \infty$.

2. Perturbations of $\exp(z)$. The fact that $E_\lambda^n(0) \to \infty$ for $\lambda = 1$ can be dramatically altered by allowing λ to be complex.

THEOREM. 1. *There exists a sequence* $\lambda(i) \to 1$ *such that* 0 *is preperiodic for* $E_{\lambda(i)}$.
2. *There exists a sequence* $\mu(i) \to 1$ *such that for all* $i \geq 3$, $E_{\mu(i)}$ *has an attracting periodic point of period* i.

COROLLARY. e^z *is not structurally stable.*

REMARK. This fact is an essential ingredient in the recent proof by Ghys, Goldberg, and Sullivan [**GGS**] that e^z is recurrent. The important question of the ergodicity of e^z remains open.

For the proof of part 1, define the family of functions $G_n(\lambda) = E_\lambda^n(0)$. This family is not normal in any neighborhood of 1. By Montel's Theorem, there exist λ-values arbitrarily close to 1 such that $G_i(\lambda) = 2k\pi i$ for any $k \in \mathbf{Z}$. But then $G_{i+1}(\lambda) = \lambda = E_\lambda(0)$ so 0 is preperiodic and λ is periodic.

The proof of part 2 is somewhat more complicated and will appear later.

3. The bifurcation diagram of E_λ. Douady and Hubbard have shown [**DH**] that the bifurcation diagram for the one-parameter family of quadratic maps $z \to z^2 + \lambda$ has a rich and interesting structure. This is the so-called Mandelbrot set (see [**Ma**]). The corresponding diagram for E_λ possesses many of the same features but also some startling differences. The results of the previous section indicate that this set should be quite complicated, at least near $\lambda = 1$.

Let $C_k = \{\lambda \in \mathbf{C} \mid E_\lambda$ has an attracting periodic orbit of period $k\}$. The C_k are disjoint open sets in the λ-plane since each E_λ can have at most one attracting periodic orbit. The proofs of the following are straightforward.

1. C_1 is bounded by a cardioid-like curve given by $\lambda = \varsigma e^{-\varsigma}$ where $|\varsigma| = 1$. Indeed, if $\lambda = \varsigma e^{-\varsigma}$, then ς is a fixed point for E_λ and, moreover, $E'_\lambda(\varsigma) = \varsigma$. Hence the eigenvalue of E'_λ at the fixed point traverses the unit circle once as λ traverses the cardioid.

2. For each nth root of unity ς_n, a component of C_n touches C_1 at $\lambda = \varsigma_n e^{-\varsigma_n}$. This component is a "tongue" which is simply connected and extends to infinity. This follows by applying the maximum principle to the eigenvalue map $\lambda \to (E^n_\lambda)'(z_n)$, where z_n is the attracting periodic point. This map omits 0 if $n > 1$.

3. The period 2 tongue contains all λ with $\lambda < -e$. The attracting period two points in this case are real. One may check that the equations $E^2_\lambda(z) = z$ and $(E^2_\lambda)'(z) = 1$ (or -1) have infinitely many solutions on the boundary of this tongue. Hence the eigenvalue traverses the unit circle infinitely often as λ traverses the boundary of this tongue. This generates infinitely many cusp-like points on this curve, as on the cardioid, as well as additional, higher period tongues emanating at roots of unity.

4. According to the results in §1, there is also a collection of curves in the λ-plane on which $J(E_\lambda) = \mathbf{C}$. These curves accumulate on $\lambda = 1$.

Figure 1 gives a computer sketch of the bifurcation diagrams for E_λ. Clearly, many questions remain. Are the C_k dense? Is there an open set of λ-values for which $J(E_\lambda) = C$?

4. The global saddle node bifurcation. Many of the bifurcations which the E_λ undergo are well understood locally. However, they are often accompanied by spectacular changes in the Julia set. We give one example: the saddle-node bifurcation.

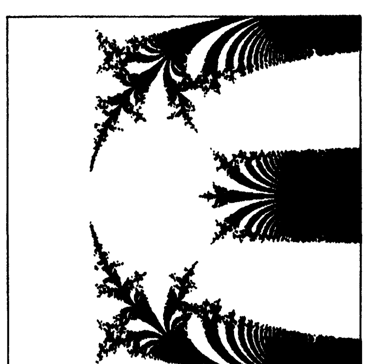

FIGURE 1. *The topological structure of the bifurcation diagram of* λe^z. Unshaded regions are the C_k's. Black regions correspond to curves where the Julia set is the whole plane. The computer algorithm used to generate this picture simply colored a point if the corresponding exponential map satisfied $\mathrm{Re}\, E^n_\lambda(0) > 100$ for some $n \leqslant 100$.

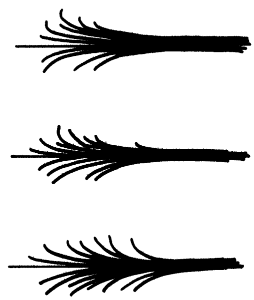

FIGURE 2. $J(E_\lambda)$ for $\lambda > 1/e$ is a Cantor set of curves.

Let $\lambda = 1/e$. For this value of λ, E_λ has a fixed point at 1 with $E'_\lambda(1) = 1$. When $\lambda > 1/e$, this fixed point separates into two repelling fixed points, one on each side of the real axis. On the other hand, when $0 < \lambda < 1/e$, E_λ has two fixed points at $q < 1 < p$ on the real axis. One checks easily that q is attracting and p is repelling. This is the usual saddle-node bifurcation in the complex plane.

The global strucure of $J(E_\lambda)$ is quite different depending upon whether $\lambda > 1/e$ or $\lambda < 1/e$.

THEOREM. 1. *If* $\lambda > 1/e$, $J(E_\lambda) = \mathbf{C}$.

2. *If* $0 < \lambda < 1/e$, $J(E_\lambda)$ *is a Cantor set of curves lying to the right of the vertical line* $x = p$.

PROOF. 1 follows from the Proposition in §1. For 2, note that the vertical line $x = p$ is mapped to the circle of radius p about the origin. Hence all points with $\operatorname{Re} z < p$ lie in the basin of attraction of q and are therefore not in $J(E_\lambda)$.

To see the Cantor set of curves, note that the preimage of the half plane $\operatorname{Re} z \geq p$ is a countable collection of parabolic regions within $\operatorname{Re} z \geq p$. Taking repeated inverse images of these sets yields a set depicted in Figure 2. Using the fact that $|E'_\lambda(z)| > 1$ if $\operatorname{Re} z > p$, it follows that this is the Julia set of E_λ.

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