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IMBEDDING ESTIMATES INVOLVING NEW NORMS AND APPLICATIONS

BY MARTIN SCHECHTER

Let $H^{s,p}$ denote the Sobolev space with norm

$$\|u\|_{s,p} = \|\overline{F}(1+|\xi|^2)^{s/2}Fu\|_p, \tag{1}$$

where F denotes the Fourier transform, $\overline{F}$ its inverse and $\|w\|_p$ is the $L^p(\mathbf{R}^n)$ norm. In many linear and nonlinear problems one comes across the question: For which functions $V(x)$ does there exist an estimate of the form

$$\|Vu\|_q \le C\|u\|_{s,p}, \qquad u \in H^{s,p}? \tag{2}$$

In this note we find a large class of functions V for which (2) holds. To do this we introduce a new family of norms $M_{\alpha,r,t,\delta}(V)$ for $0 \le \alpha$, $0 < \delta < 1$, $1 \le r < \infty$ and $1 \le t \le \infty$. For $x \in \mathbf{R}^n$ let

$$\begin{aligned} \omega_\alpha(x) &= |x|^{\alpha-n}, && 0<\alpha<n,\\ &= 1-\log|x|, && \alpha = n,\\ &= 1, && n<\alpha. \end{aligned}$$

When $0 < \alpha$ we define

$$\begin{aligned} M_{\alpha,r,t,\delta}(V) &= \left(\int\left(\int_{|x-y|<\delta}|V(x)|^r\omega_\alpha(x-y)\,dx\right)^{t/r}dy\right)^{1/t}, \qquad 1\le t<\infty,\\ &= \sup_y\left(\int_{|x-y|<\delta}|V(x)|^r\omega_\alpha(x-y)\,dx\right)^{1/r}, \qquad t=\infty. \end{aligned}$$

For $\alpha = 0$ we put

$$M_{0,r,t,\delta}(V) = \|V\|_t.$$

We also set

$$M_{\alpha,r,t}(V) = M_{\alpha,r,t,1}(V)$$

and make the following basic assumption.

HYPOTHESIS A. The parameters α, r, s, t, p, q satisfy:

A1. $0 \le \alpha, s$; $1 \le q \le r < \infty$; $1 \le p < \infty$; $1 \le t \le \infty$.

A2. $1/q \le 1/p + 1/t$.

A3. $\alpha/nr \le s/n + 1/q - 1/p - 1/t$.

A4. We do not have both $q = t$ and $n = sp$ or both $p = 1$ and $s/n = 1/t + 1/q'$.

Let $M_{\alpha,r,t}$ denote the set of those functions $V(x)$ such that $M_{\alpha,r,t}(V) < \infty$.

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THEOREM 1. *Under Hypothesis* (A) *assume that:*
(a) *If* $q < r$ *and* $s < n$, *then either* $p \neq 1$ *or inequality* A3 *is strict.*
(b) *If* $q = r$ *and* $s < n$, *then either* $r < t \leq p'$ *or inequality* A3 *is strict.*
Let $V(x)$ *be a function in* $M_{\alpha,r,t}$. *Then multiplication by* V *is a bounded operator from* $H^{s,p}$ *to* L^q. *There is a constant* C_0 *depending only on the parameters such that*

$$\|Vu\|_q \leq C_0 M_{\alpha,r,t}(V)\|u\|_{s,p}, \qquad u \in H^{s,p}. \tag{3}$$

Moreover, there are constants C_1, C_2 *depending only on the parameters such that*

$$\|Vu\|_q \leq C_1 M_{\alpha,r,t,\delta}(V)\|u\|_{s,p} + C_2 M_{\alpha,r,t}(V)\|u\|_p, \qquad u \in H^{s,p}, \tag{4}$$

and C_1 *does not depend on* δ.

COROLLARY 2. *If, in addition,*

$$M_{\alpha,r,t,\delta}(V) \to 0 \quad \text{as } \delta \to 0 \tag{5}$$

then for every $\epsilon > 0$ *there is a constant* K *such that*

$$\|Vu\|_q \leq \epsilon\|u\|_{s,p} + K\|u\|_p, \qquad u \in H^{s,p}. \tag{6}$$

If $t \neq \infty$, *then multiplication by* V *is a compact operator from* $H^{s,p}$ *to* L^q. *The same conclusion holds when* $t = \infty$ *if*

$$\int_{|x-y|<1} |V(x)|^r \omega_\alpha(x-y)\, dx \to 0 \quad \text{as } |y| \to \infty. \tag{7}$$

THEOREM 3. *Let* $\psi(\rho)$ *be a positive function such that* $\int_0^1 \psi(\rho)\rho^{-1}\, d\rho < \infty$. *Under Hypothesis* A, *inequalities* (3) *and* (4) *hold without the restrictions* (a) *and* (b) *of Theorem* 1 *provided we replace* $\omega_\alpha(x)$ *by* $\omega_\alpha(x)\psi(x)^{-b}$ *in the definition of* $M_{\alpha,r,t}(V)$, *where* $b = r(1 + 1/q - 1/p - 1/r - 1/t)$.

We apply these results to a nonlinear problem. Let m be a positive integer, Ω an arbitrary (bounded or unbounded) domain $\mathbf{R}^n$ and let $W = H_0^{m,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in $H^{m,2}$. Let $a(u,v)$ be a hermitian bilinear form on W satisfying

$$K_1^{-2}\|u\|_{m,2}^2 \leq a(u) = a(u,u) \leq C^2\|u\|_{m,2}^2, \qquad u \in W.$$

Let $f(x,v)$, $g(x,v)$ be continuous functions on $\Omega \times \mathbf{R}$ such that $f(x,v) = \partial g(x,v)/\partial v$. We assume that

$$g(x,v) \leq B(x,v) = \sum_{k=1}^{N} V_k(x)|v|^{q_k}, \qquad x \in \Omega,\ v \in \mathbf{R}, \tag{8}$$

and

$$M_{\alpha_k,r_k,t_k,\delta}(V_k) \to 0 \quad \text{as } \delta \to 0, \qquad 1 \leq k \leq N,$$

where

$$1 < q_k/2 + 1/t_k, \qquad \alpha_k/nr_k \leq mq_k/n + 1 - q_k/2 - 1/t_k.$$

If $t_k = \infty$, we assume that (7) holds for V_k, α_k, r_k. By Theorem 1 there are constants M_k such that

$$\int_\Omega B(x, u(x))\, dx \le \sum_{k=1}^{N} M_k \|u\|_{m,2}^{q_k}, \qquad u \in W. \tag{9}$$

Assume that

$$K_2 = \int_\Omega g(x, 0)\, dx$$

exists, and put

$$M(R) = R^{-2}\left(\sum_1^N M_k (K_1 R)^{q_k} - K_2\right), \qquad \lambda_0^{-1} = \inf_{0<R} M(R).$$

Let A be the operator associated with $a(u, v)$ (cf. [8]), and let $\lambda > 0$. We are looking for a solution of

$$Au = \lambda f(x, u). \tag{10}$$

THEOREM 4. *If $\lambda < \lambda_0$, then, for any R such that $\lambda M(R) < 1$, (10) has a solution satisfying $a(u) \le R^2$.*

There is a connection between the spaces $M_{\alpha,r,t}$ and the Lorentz spaces $L^{\sigma,t}$ (for the definitions cf. [**3**, **11**, **13**]).

THEOREM 5. *If $0 \le 1/\sigma - 1/t = \alpha/nr$, $r < \sigma \le t < \infty$, then*

$$M_{\alpha,r,t}(V) \le C\|V\|_{L^{\sigma,t}}, \qquad V \in L^{\sigma,t}. \tag{11}$$

If we combine this with Theorem 1 we obtain

THEOREM 6. *If $t < \infty$ and $1/q - 1/p \le 1/t \le 1/\sigma < 1/q$, $1/\sigma + 1/p \le s/n + 1/q$, then*

$$\|Vu\|_q \le C\|V\|_{L^{\sigma,t}}\|u\|_{s,p}. \tag{12}$$

Special cases of inequality (2) were proved by Stummel [**12**], Balslev [**2**], Berger-Schechter [**4**] and Schechter [**7**, **8**, **9**]. Our solution of (10) avoids some of the hypotheses of Noussair-Swanson [**6**]. The suggestion that there should be an inequality of the form (12) is due to H. Brezis.

Theorem 1 is proved by using Bessel potentials as investigated by Aronszajn-Smith [**1**]. Inequality (3) is equivalent to

$$|(G_s * f, Vu)| \le C\|f\|_p\|v\|_{q'} \times\left(\int\left(\int |V(x)|^r G_s(x-y)^r \phi(x-y)^{-r}\, dx\right)^{t/r} dy\right)^{1/t}$$

holding for a suitably chosen function ϕ. This is derived by tedious and tricky estimates. Corollary 2 follows by standard arguments and Theorem 3 is a slight variation. Theorem 4 is proved by variational techniques using (8) and (9). Theorem 5 is proved by using the fact that $H^{s,p} \subset L^q$ for certain values of s, p, q. By real interpolation we find that $H^{s,p} \subset L^{\sigma,p}$ for specific values. This leads to (11). Theorem 6 is merely a combination of Theorems 1 and 5.

REFERENCES

1. N. Aronszajn and K. T. Smith, *Theory of Bessel potentials*, Ann. Inst. Fourier Grenoble **11** (1961), 385–475.

2. E. Balslev, *The essential spectrum of elliptic differential operators in $L^p(R_n)$*, Trans. Amer. Math. Soc. **116** (1965), 193–217.

3. J. Bergh and J. Lofstrom, *Interpolation spaces*, Springer-Verlag, Berlin, 1976.

4. M. S. Berger and M. Schechter, *Embedding theorems and quasilinear elliptic boundary value problems for unbounded domains*, Trans. Amer. Math. Soc. **172** (1972), 261–278.

5. K. Jorgens and J. Weidmann, *Spectral properties of Hamiltonian operators*, Lecture Notes in Math., vol. 313, Springer-Verlag, Berlin, 1973.

6. E. S. Noussair and C. A. Swanson, *Existence theorems for generalized Klein-Gordon equations*, Bull. Amer. Math. Soc. **8** (1983), 333–336.

7. M. Schechter, *On the invariance of the essential spectrum of an arbitrary operator*. II, Ricerche Mat. **16** (1967), 3–26.

8. ____, *Spectra of partial differential operators*, North-Holland, Amsterdam, 1971.

9. ____, *Essential self-adjointness of the Schrödinger operator with magnetic vector potential*, J. Funct. Anal. **20** (1975), 93–101.

10. B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. **7** (1982), 447–526.

11. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971.

12. F. Stummel, *Singuläre elliptische Differentialoperatoren in Hilbertschen Räumen*, Math. Ann. **132** (1956), 150–176.

13. H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK, NEW YORK 10012

Current address: Department of Mathematics, University of California, Irvine, California 92717