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Group extensions, representations, and the Schur multiplicator, by F. Rudolf Beyl and Jürgen Tappe, Lecture Notes in Math., Vol. 958, Springer-Verlag, Berlin, 1982, iv + 278 pp., \$13.50. ISBN 3-5401-1954-X

Schur multipliers arise when one studies central extensions of groups. A central extension is a surjective homomorphism $\varphi: G \to Q$ whose kernel is contained in the center of G. One also calls G itself a central extension of Q. Schur was interested in finding all projective representations of a given finite group Q, i.e. all homomorphisms $\rho: Q \to \mathrm{PGL}_n(\mathbf{C})$ with $n \geqslant 2$. The group $\mathrm{PGL}_n(\mathbf{C})$ comes with a central extension $\pi: \mathrm{GL}_n(\mathbf{C}) \to \mathrm{PGL}_n(\mathbf{C})$, where π is the usual map associating with a linear transformation of $\mathbf{C}^n$ an automorphism of projective $n - 1$ space ($n \geqslant 2$). The kernel of π is the center of $\mathrm{GL}_n(\mathbf{C})$ and may be identified with $\mathbf{C}^* = \mathrm{GL}_1(\mathbf{C})$. Pulling back π along ρ one gets a central extension $\varphi: G \to Q$ with kernel $\mathbf{C}^*$ and the situation is that of Diagram 1.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & Q \\ \downarrow \sigma & & \downarrow \rho \\ \mathrm{GL}_n(\mathbf{C}) & \xrightarrow{\pi} & \mathrm{PGL}_n(\mathbf{C}) \end{array}$$

DIAGRAM 1

Thus we have associated with the projective representation ρ of Q the linear representation σ of G. Conversely, if $\varphi: G \to Q$ is any central extension and $\sigma: G \to \mathrm{GL}_n(\mathbf{C})$ is an irreducible linear representation, one obtains by Schur's Lemma a projective representation ρ of Q such that Diagram 1 commutes. Schur discovered [7, 1902] that there is at least one finite central extension $\varphi: G \to Q$ such that σ exists for all ρ (n may vary), i.e. such that the projective representations of Q all come from linear representations of G. If one knows G, one may classify its linear representations by character theory. Of course one takes G minimal here. Then Schur calls G a representation group of Q (Darstellungsgruppe). As this term no longer sounds like what it is trying to convey, let us say instead that $\varphi: G \to Q$ is a Schur extension of Q. In general there is no unique Schur extension of Q, but Schur discovered that the kernel $M(Q)$ of φ is unique (up to canonical isomorphism). He baptized it the multiplier of Q (Multiplikator). Unfortunately he also called $H^2(Q, \mathbf{C}^*)$ the multiplier and identified it with $M(Q)$ by observing that the character group $\mathrm{Hom}(A, \mathbf{C}^*)$ of a finite abelian group A (such as $M(Q)$) is isomorphic with A.

This isomorphism is not canonical however. There is now some confusion as to what the term "Schur multipliers" refers to. The usual meaning is described by the formula $M(Q) = H_2(Q, \mathbf{Z})$, which is also used when Q is infinite. The competing candidate is its dual $\mathrm{Hom}(H_2(Q,\mathbf{Z}), \mathbf{C}^*) \simeq H^2(Q, \mathbf{C}^*)$, which may be quite different. Schur carried out his program (to find all projective representations) for several finite groups, in particular the symmetric and the alternating groups [9]. In the case of a symmetric group on at least 4 letters, there are two Schur extensions, but for alternating groups the Schur extension is unique. This is because the alternating groups A_n are perfect ($n \geqslant 4$), i.e. equal to their own commutator subgroup. For a perfect group Q the (unique) Schur extensions coincides with the universal central extension $\tilde{Q} \to Q$, which gets its name from the following property. For any central extension $G \to Q$ the homomorphism $\tilde{Q} \to Q$ factors uniquely as $\tilde{Q} \to G \to Q$. This characterisation is often helpful when studying the multiplier of a perfect group, e.g. a simple group. For all finite simple groups the multiplier has been determined, presumably [3]. As central extensions of simple groups occur naturally in the study of the internal structure of finite simple groups, knowledge of the Schur multipliers is one of the many ingredients going into the proof of the classification of finite simple groups. Steinberg's work on the multipliers of finite Chevalley groups (or finite simple groups of Lie type) led Milnor to his definition of K_2 of a ring as being the Schur multiplier of the commutator group $E(R)$ of $\mathrm{GL}(R) = \lim_n \mathrm{GL}_n(R)$. The group $E(R)$ is perfect, so that one may study $K_2(R)$ by investigating the universal central extension $\mathrm{St}(R)$ of $E(R)$ (the Steinberg group [6]). The group $K_2(R)$ is already quite interesting when R is a field. It is then generated by so-called symbols (related to symbols in number theory) and has recently been used by Merkurjev and Suslin to solve an old problem in the theory of Brauer groups. (Much more is involved in this work. See Soulé [10].)

Schur gave yet another description of his multiplier, using a presentation $1 \to R \to F \to Q \to 1$ of the finite group Q [8, 1907]. Here F is a free group and the group R of relators is the kernel of the surjective homomorphism $F \to Q$. One forms the mixed commutator subgroup $[F, R]$ of F generated by the commutators $xyx^{-1}y^{-1}$ with $x \in F$, $y \in R$. Intersecting $R/[F, R]$ with the commutator subgroup of $F/[F, R]$, one gets the multiplier. The formula

$$M(Q) = [F, F] \cap R/[F, R]$$

is nowadays called Hopf's formula, because of Hopf's influential paper in 1942 [4]. In this paper Hopf shows that for any group Q his formula defines an abelian group which depends only on Q, not on the chosen presentation. For finite groups this is Schur's result, of which Hopf was apparently not aware. More importantly, Hopf established an exact sequence

$$\pi_2(X) \to H_2(X, \mathbf{Z}) \to M(\pi_1(X)) \to 0,$$

where X is a complex. (Later it was checked that X may be replaced by a more general topological space.) In particular, if the second homotopy group $\pi_2(X)$ vanishes, the sequence identifies the Schur multiplier of the fundamental group $\pi_1(X)$ with the second homology group of X, thus giving a topological interpretation of the multiplier. This connection between algebra and topology suggested how to define homology and cohomology groups of a group Q in arbitrary dimension and immensely stimulated the development of homological algebra. For a nice discussion of the impact of Hopf's paper, read Mac Lane [5].

In our present understanding of Schur multipliers and Schur extensions, certain exact sequences are crucial. For instance, if φ: $G \to Q$ is a central extension with kernel A, one has the following exact sequence which comes as a byproduct of the Hochschild-Serre spectral sequence (its exactness may also be proved directly):

$$0 \to \mathrm{Hom}(Q, \mathbf{C}^*) \to \mathrm{Hom}(G, \mathbf{C}^*) \to \mathrm{Hom}(A, \mathbf{C}^*)$$
$$\to H^2(Q, \mathbf{C}^*) \to H^2(G, \mathbf{C}^*).$$

The Schur extensions are those for which the map $\mathrm{Hom}(A, \mathbf{C}^*) \to H^2(Q, \mathbf{C}^*)$ is an isomorphism. The existence of σ in Diagram 1 may be explained from the vanishing of the last map in the sequence. Here one uses the well-known intepretation of the second cohomology group in terms of extensions. Given a Q-module A there is a bijective correspondence between $H^2(Q, A)$ and the set, say Opext(Q, A), of isomorphism classes of extensions of Q by A. Here an extension of Q by A is a surjective homomorphism φ: $G \to Q$ with kernel A so that the action of G on A by inner conjugation agrees via φ with the given action of Q on A. As $A \mapsto H^2(Q, A)$ is an additive functor, the same must be true of $A \mapsto$ Opext(Q, A), which is not immediate from its definition. It is a general principle that in order to understand additive structure on a category one must look at direct sums. Thus in order to understand the additive behaviour of Opext(Q, A) without recourse to cocycles, we ask why Opext(Q, $A \oplus B$) is the product of Opext(Q, A) and Opext(Q, B). The answer is given by forming fibred products over Q of extensions (easy exercise). It is also easy to see how a surjective homomorphism α: $A \to B$ of Q-modules induces a map from Opext(Q, A) to Opext(Q, B). Given an extension of Q by A one simply factors out $\ker \alpha$ and identifies $A/\ker \alpha$ with B. After this preparation we are ready to discover the Baer sum, i.e. the rule of addition in Opext(Q, A). Namely, addition is the map from Opext(Q, A) $\times$ Opext(Q, A) to Opext(Q, A) which corresponds to the map from Opext(Q, $A \oplus A$) to Opext(Q, A) induced by the surjective addition homomorphism $A \oplus A \to A$. Working this out one finds that I have dissected the description given by Baer in 1934 [1]. To finish the exercise one may figure out what an arbitrary homomorphism α: $A \to B$ of Q-modules does to Opext by recalling how the graph of a function is explained: Factor α into a surjection $A \oplus B \to B$ and the standard inclusion $A \to A \oplus B$.

What about the book? It is largely expository. The exposition is overexpanded. It starts out with a treatment of things like Opext from the wrong point of view. Namely, the authors try to cut their subject loose from the homological algebra and the topology that are so crucial for our present understanding. Of course it is feasible to present proofs in which a two-dimensional cocycle is still called a factor set, a one-dimensional cocycle a derivation and $H^2(Q, A)$ is baptized $H(Q, A, \varphi)$ (sic!). However, I fail to see where the simplification is. Homological algebra presents a very simple entry to the subject and is, moreover, indispensible for a lot of important mathematics. It is true that one should figure out what abstract notions mean in a more concrete context, as in the case of the Baer sum. But one expects in a Springer Lecture Note that something more than that has happened after 66 pages. Another complaint concerns the title. It suggests that one will learn a lot about representation theory, not just the fact that Schur mulipliers occur when one tries to lift a projective representation to a linear one. It would have made more sense to put "isoclinism" in the title. If you want to learn about isoclinism and its connections with the multiplier, I suggest reading instead the survey article by one of the authors [2]. It has a swifter pace so that one can see more easily where things are going. In the same proceedings one may also read Wiegold's paper on the Schur multiplier. On page 204 of the book one should modify the definition of unicentral to save what follows. The correct requirement is that $\pi Z(G)$ equals $Z(Q)$ for all central extensions $\pi\colon G \to Q$.

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Introduction to number theory, by Loo Keng Hua, Springer-Verlag, Berlin, 1982, xviii + 572 pp., $46.00. ISBN 0-3871-0818-1

The book under review is a translation from the Chinese of a book first published in 1956. (The Chinese edition was reviewed by K. Mahler in *Mathematical Reviews*.) Some of the chapters in this translation have been