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Notes on real and complex C^*-algebras, by. K. R. Goodearl, Shiva Mathematics Series, Shiva Publishing, Ltd., Cambridge, Mass., 1982, 211 pp., $28.95. ISBN 0-9068-1216-X

The author of this book states his purpose clearly: "W[e] have tried to present to the non-specialist a view into the subject by means of its most striking theorems." He does not even hint at the vast range of the subject but merely covers a few things well. In a similar vein this review is directed not at the expert but at those who want to know why so many mathematicians study (and write books about) C^*-algebras. The review will follow the book in requiring C^*-algebras to be unital (i.e. to have multiplicative identity elements). It will also assume complex scalars except to discuss the real case.

The story begins with two classes of concrete examples: one commutative and the other not. Let S be a compact Hausdorff space. Let $C(S)$ be the space of all continuous complex valued functions on S. Under pointwise addition and multiplication $C(S)$ is a commutative algebra. That is, it is a linear space which is also a commutative ring under the same additive structure and with the scalar and ring multiplications agreeing as one would expect. In addition $C(S)$ has two other elements of structure which turn out to be crucial to its study. It has a norm (the supremum or uniform norm) defined by

$$\|f\| = \sup\{|f(s)| : s \in S\} \quad \forall f \in C(S)$$

under which it is a complete normed algebra or Banach algebra. It also has an involution $*$: $C(S) \to C(S)$ defined by

$$f^*(s) = f(s)^- \quad \forall f \in C(S), \forall s \in S,$$

where the bar denotes complex conjugation. We will say more about involutions below, but for now we remark that an algebra with a (fixed) involution is called a $*$-algebra (pronounced star-algebra) and a Banach algebra with an involution is called a Banach $*$-algebra.

Now suppose S_1 and S_2 are both compact Hausdorff spaces and ϕ: $S_1 \to S_2$ is a continuous function. Then one may define $C(\phi)$: $C(S_2) \to C(S_1)$ by

$$C(\phi)(f): f \circ \phi, \qquad f \in C(S_2).$$

It is obvious that $C(\phi)$: $C(S_2) \to C(S_1)$ preserves the structure introduced above so that it is a $*$-homomorphism (i.e. a linear ring homomorphism satisfying $C(\phi)(f^*) = (C(\phi)(f))^*$) which is also contractive: $\|C(\phi)(f)\| \leq \|f\|$. It is also easy to see that $C(\cdot)$ is a contravariant functor on the category of compact Hausdorff spaces and continuous maps. It is not so easy to see, but it is true, that $C(\cdot)$ is an anti-isomorphism of categories. Thus every theorem about compact Hausdorff spaces and continuous functions is equivalent to a theorem about this special class of commutative Banach $*$-algebras and contractive $*$-homomorphisms.

We turn to the second class of examples. Let H be a Hilbert space and let $B(H)$ be the space of all continuous linear maps T: $H \to H$. Now $B(H)$ is an algebra under pointwise linear operations and composition as a product. It is also a Banach algebra under the operator norm

$$\|T\| = \sup\{\|T(x)\|/\|x\| : x \in H, x \neq 0\}$$

and a Banach $*$-algebra under the Hilbert space adjoint (defined by $(T^*x, y) = (x, Ty)$ for all $x, y \in H$ where $(\cdot, \cdot)$ denotes the inner product of H) as involution. The class of examples we wish to consider consists of any unital subalgebra of $B(H)$ which is closed under the involution and closed in the norm topology. A subalgebra closed under the involution is called a $*$-subalgebra.

It turns out that these two classes of examples share one more property, called the C^*-condition:

$$\|a^*a\| = \|a\|^2.$$

A Banach $*$-algebra satisfying the C^*-condition is called a C^*-algebra. The first section of the book under review contains standard, neat proofs of two theorems called the commutative and the general Gelfand-Naimark structure theorems. These theorems assert that: (1) Every commutative C^*-algebra is canonically isometrically $*$-isomorphic to $C(S)$ for some compact space S; and (2) Every (not necessarily commutative) C^*-algebra is isometrically $*$-isomorphic to some subalgebra of some $B(H)$ closed under both the norm and the involution. These are remarkably pleasing results. A very short list of axioms is seen to characterize two important classes of concrete examples completely. It is even more striking that the commutative case encompasses (in the sense of category theory) the topological theory of compact Hausdorff spaces. It turns out that in a very deep sense C^*-algebra theory is the study of "noncommutative topology". Since this idea transcends the scope of the book under review we will not pursue it.

Perhaps it is already clear why one might be interested in C^*-algebras. However there is an even more basic reason. Most algebraic objects fit neatly into well-behaved categories which expose the most fundamental properties of the objects. Unfortunately the objects of analysis, which are generally topological-algebraic in nature, often fit awkwardly into their categories. These categories are less well behaved and sometimes mask, rather than expose, the fundamental properties of the analytic objects. However, C^*-algebras are immune from these problems because of the following fundamental theorem.

THEOREM. *Let A and B be C^*-algebras and let ϕ: $A \to B$ be a $*$-homomorphism. Then ϕ is contractive, $\phi(A)$ is a closed $*$-subalgebra of B (and hence a C^*-algebra again), and the natural map $\tilde{\phi}$: $A/\mathrm{Ker}\,(\phi) \to \phi(A)$ is an isometry.*

As a result of this theorem we can define the morphisms in the category of C^*-algebras to be $*$-homomorphisms without any continuity restriction. Hence the category is a well-behaved "algebraic" category.

Looked at more deeply this theorem suggests that the complete norm is almost superfluous in C^*-algebras. Almost, but not quite. To get the nice properties of C^*-algebras one must postulate the existence of such a norm. The theorem shows that a complete norm satisfying the C^*-condition is unique when it exists. Thus, in principle, the norm can be derived from the $*$-algebraic structure. It turns out that there are many ways to do this, of which we will mention four.

(1) For each $a \in A$ let $\rho(a) = \sup\{|\lambda|: \lambda 1 - a \text{ is not invertible in } A\}$. Then

$$\|a\| = \rho(a^*a)^{1/2}.$$

(2) $$\|a\| = \sup\{\|\phi(a)\|: \phi \text{ is a } *\text{-homomorphism of } A \text{ into } B(H) \text{ for some Hilbert space } H\}.$$

(3) Let $A_U = \{u \in A: uu^* = u^*u = 1\}$. Then

$$\|a\| = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : a = \sum_{j=1}^{n} \lambda_j u_j, \lambda_j \in \mathbf{C}, u_j \in A_U\right\}.$$

(4) Let $A_+ = \{\sum_{j=1}^{n} a_j^* a_j: a_j \in A\}$. Then

$$\|a\| = \inf\{t \geqslant 0: t^2 1 - a^*a \in A_+\}.$$

Except for (1) the proof of any of these formulas requires considerable work. Each of the formulas implies the first conclusion of the theorem almost trivially. The rest of the theorem follows from the commutative theory.

In addition to the above fundamental results, many others are known which point to the very special character of C^*-algebras among all infinite dimensional algebras. C^*-algebras may be studied as algebras, $*$-algebras, Banach spaces, Banach algebras, or in terms of their order structure. These elements of structure are bound together very tightly so that various subsets completely determine the whole.

These remarks show how extremely well behaved C^*-algebras are. They are also important. However in the reviewer's opinion C^*-algebras have sometimes been studied because of their good properties, when other objects deserved more attention. For instance, consider a locally compact topological group G, and $L^1(G)$ its usual group algebra under convolution multiplication. The group is determined up to homeomorphic isomorphism by the Banach algebra structure of $L^1(G)$. Like any Banach $*$-algebra, $L^1(G)$ has an enveloping C^*-algebra $C^*(G)$. Many different groups G give rise to isomorphic C^*-algebras $C^*(G)$. Thus the passage from $L^1(G)$ to $C^*(G)$ destroys much useful information. At present $C^*(G)$ receives much more attention than $L^1(G)$. Eventually the emphasis needs to shift.

We have discussed the subject of the first section of Goodearl's book, which contains results that can be found in many other books. The other two sections contain results not so readily available. The second section deals with real C^*-algebras and the third with AF-algebras. There is no other complete exposition of real C^*-algebras available in a book. This reflects in part the fact that no use has yet been found for real C^*-algebras. Indeed at the end of a study of real C^*-algebras one finds that complexifying (i.e. taking the real tensor product with the complex numbers) completely preserves their representation theory. Despite this final result there is no known way to derive the real results (such as the real analogues of the two theorems of Gelfand and Naimark included in Goodearl's book) from the complex case without very considerable work. The problem is that it is hard to find a workable formula for the C^*-norm in the complexification. Goodearl therefore derives the real case essentially independently. It seems likely that eventually an ingenious, easy way will be found to derive the real from the complex theory using one of the formulas for the norm described above. The reviewer also believes that the placement of various real C^*-algebras into a given complex C^*-algebra may eventually be seen to hold some interest.

The third part of the book deals with AF-algebras. These were introduced in 1972 by Bratteli [**1**] as direct limits (in the category of C^*-algebras defined above) of a sequence of finite dimensional C^*-algebras. In his original paper Bratteli characterised them as those C^*-algebras which are approximately finite dimensional (hence the name AF-algebra) in the following precise sense.

THEOREM. *A separable C^*-algebra A is an AF-algebra if and only if for every finite set $a_1, a_2, \ldots, a_n$ in A and every $\varepsilon > 0$ there exists a finite dimensional C^*-subalgebra B of A and elements $b_1, b_2, \ldots, b_n$ of B satisfying $\|b_j - a_j\| < \varepsilon$ for $j = 1, 2, \ldots, n$.*

The AF-algebras are remarkably diverse in their other properties. Since they are relatively easy to work with, they have provided many important examples.

Bratteli originally studied AF-algebras by means of diagrams (now called Bratteli diagrams). He showed that the diagrams determine the limiting AF-algebras up to isomorphism and that, for instance, the ideal structure of the limiting algebra could be read off the diagram. However, it was clear that quite different diagrams could give rise to isomorphic AF-algebras and no method was known to determine when this happened. In 1976 Elliott [**5**] introduced a rather complicated invariant, called the range of the dimension, which overcame this difficulty. This invariant consisted of equivalence classes of idempotents in the algebra, where e and f are equivalent if there exist elements a and b

with $ab = e$, $ba = f$. (In a full matrix ring two idempotents are equivalent in this sense if and only if they have the same dimension.) There was also additional structure. As K-theory began to be used more for C^*-algebras, Elliott's invariant was naturally reformulated in terms of K_0.

For any ring A with identity, $K_0(A)$ is a group consisting of differences of equivalence classes of idempotents. In order to make $K_0(A)$ into a group it is necessary to use idempotents in all the matrix rings over A at once. This allows one to define the sum $p \oplus q$ of two idempotents $p, q \in A$ as $\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$ in the 2×2 matrix ring over A, and so on. In order to achieve the cancellation law in the semigroup of equivalence classes, one needs to define a larger equivalence relation than that introduced above: p and q are equivalent if there is any idempotent r so that $p \oplus r$ is equivalent to $q \oplus r$ as defined above. If $A = C(S)$ is a commutative C^*-algebra then $K_0(A)$ coincides with the topological invariant $K^0(S)$ defined in terms of vector bundles.

Elliott's work shows that $K_0(A)$ is an ordered group when A is an AF-algebra. As an ordered group $K_0(A)$ almost determines A up to isomorphism. In fact it determines $A \otimes K$ up to isomorphism, where K is the C^*-algebra of compact operators. In order to remove the tensor product with K, it is necessary to keep track of the equivalence class $[1_A]$ of the identity element in A also. This gives the following fundamental result.

THEOREM. *Let A and B be AF-algebras. Then A and B are isomorphic as C^*-algebras if and only if $K_0(A)$ and $K_0(B)$ are isomorphic as ordered groups under an isomorphism taking* $[1_A]$ *onto* $[1_B]$.

This theorem is of great value since $K_0(A)$ can be explicitly calculated in many cases.

All the above results are efficiently proved in Goodearl's book. In addition he characterizes those ordered groups which arise as $K_0(A)$ for some AF-algebra A. This result is due to Effros, Handelman and Shen in 1980 [**4**].

Goodearl's book was developed for a one semester introductory course in C^*-algebras. It could be used for such a course which would be accessible to most second year graduate students. Each of its 21 sections is followed by a small number of exercises. Since this book is only a brief introduction to the subject, a short list of standard references is appended: Dixmier [**2**], Kadison and Ringrose [**6**], Pedersen [**7**] and Takesaki [**8**]; and for those interested in additional information on AF-algebras, the small book by Effros [**3**].

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Group extensions, representations, and the Schur multiplicator, by F. Rudolf Beyl and Jürgen Tappe, Lecture Notes in Math., Vol. 958, Springer-Verlag, Berlin, 1982, iv + 278 pp., $13.50. ISBN 3-5401-1954-X

Schur multipliers arise when one studies central extensions of groups. A central extension is a surjective homomorphism $\varphi\colon G \to Q$ whose kernel is contained in the center of G. One also calls G itself a central extension of Q. Schur was interested in finding all projective representations of a given finite group Q, i.e. all homomorphisms $\rho\colon Q \to \mathrm{PGL}_n(\mathbf{C})$ with $n \geqslant 2$. The group $\mathrm{PGL}_n(\mathbf{C})$ comes with a central extension $\pi\colon \mathrm{GL}_n(\mathbf{C}) \to \mathrm{PGL}_n(\mathbf{C})$, where π is the usual map associating with a linear transformation of $\mathbf{C}^n$ an automorphism of projective $n-1$ space ($n \geqslant 2$). The kernel of π is the center of $\mathrm{GL}_n(\mathbf{C})$ and may be identified with $\mathbf{C}^* = \mathrm{GL}_1(\mathbf{C})$. Pulling back π along ρ one gets a central extension $\varphi\colon G \to Q$ with kernel $\mathbf{C}^*$ and the situation is that of Diagram 1.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & Q \\ \downarrow \sigma & & \downarrow \rho \\ \mathrm{GL}_n(\mathbf{C}) & \xrightarrow{\pi} & \mathrm{PGL}_n(\mathbf{C}) \end{array}$$

DIAGRAM 1

Thus we have associated with the projective representation ρ of Q the linear representation σ of G. Conversely, if $\varphi\colon G \to Q$ is any central extension and $\sigma\colon G \to \mathrm{GL}_n(\mathbf{C})$ is an irreducible linear representation, one obtains by Schur's Lemma a projective representation ρ of Q such that Diagram 1 commutes. Schur discovered [7, 1902] that there is at least one finite central extension $\varphi\colon G \to Q$ such that σ exists for all ρ (n may vary), i.e. such that the projective representations of Q all come from linear representations of G. If one knows G, one may classify its linear representations by character theory. Of course one takes G minimal here. Then Schur calls G a representation group of Q (Darstellungsgruppe). As this term no longer sounds like what it is trying to convey, let us say instead that $\varphi\colon G \to Q$ is a Schur extension of Q. In general there is no unique Schur extension of Q, but Schur discovered that the kernel $M(Q)$ of φ is unique (up to canonical isomorphism). He baptized it the multiplier of Q (Multiplikator). Unfortunately he also called $H^2(Q, \mathbf{C}^*)$ the multiplier and identified it with $M(Q)$ by observing that the character group $\mathrm{Hom}(A, \mathbf{C}^*)$ of a finite abelian group A (such as $M(Q)$) is isomorphic with A.