hypotheses is introduced, but it is not followed dogmatically. Instead, the treatment centers on four basic "principles": equilibrium, maximum, polarity, and energy. Each is discussed under alternative hypotheses and from different points of view.

This is sound pedagogy. It does little good to proceed in such matters as if there were a single best set of extra assumptions. But it does not disguise the fact that doing potential theory probabilistically can lead to complications. A famous probabilist was once heard to say that studying Hunt-style potential theory is a good way to grow old before one's time, and there is no doubt a grain of truth in the remark. The present book, however, succeeds to a remarkable degree in rejuvenating the subject. In any case one cannot cease to marvel at the dexterity with which its author walks the highwire between probability and analysis.

FRANK B. KNIGHT

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Combinatorial integral geometry with applications to mathematical stereology, by R. V. Ambartzumian, John Wiley & Sons, Somerset, New Jersey, 1982, xvii + 221 pp., $45.00. ISBN 0-4712-7977-3

In 1890 J. J. Sylvester, still creative at age 76, published a paper [7] entitled *On a funicular solution of Buffon's "problem of the needle" in its most general form*. The work is headed by the phrase "quaintly made of cords" from *The Two Gentlemen of Verona*, and is replete with well-executed drawings resembling complicated block and tackle devices. Although Sylvester's ropes serve him better than those of Shakespeare's Valentine, the mathematics suffers from the vagueness ever present in early writings on measure and probability.

To get a rough idea of the type of problem considered by Sylvester, suppose that a number of needles of various lengths are welded into a fixed planar configuration and then are "tossed at random" onto a plane ruled by equally spaced parallel lines. Calculate the probability that some single line will cross all of the needles.

The present book, *Combinatorial integral geometry*, by R. V. Ambartzumian, places the work of Sylvester in a rigorous setting while broadly extending the key idea in his paper. The main body of this book is based entirely on topics taken from the author's works; the order of the chapters seems to be generally in the chronological order of appearance of the corresponding research. Before proceeding, let us introduce a bit of formalism, simple by hindsight today, but denied Sylvester.

In E^2 an oriented line is determined by a unit normal $\mathbf{u}$ and a signed distance S to the origin. Thus $S^1 \times R$, conveniently realized as the cylinder $r = 1$ in E^3, is established as a natural coordinate system for the oriented lines of E^2. The usual surface measure on the cylinder transfers as a motion invariant measure on the oriented linesets. By identifying the points $(1, \theta, t)$ and $(1, \theta + \pi, -t)$, one obtains a coordinatization of the unoriented lines of E^2 as well as an invariant measure μ on the linesets. If d is the euclidean metric on E^2 and g is a typical line, the fundamental Cauchy-Crofton formula for the length of a segment $\overline{pq}$ states that

$$2d(p, q) = \mu[g: g \cap \overline{pq} \neq \varnothing]. \tag{1}$$

For further information on the history of formula (1), see the book by L. Santaló [6], which contains a vest bibliography.

To formulate in this setting a problem that is essentially equivalent to the previously mentioned problem of Sylvester, suppose that n line segments are arranged in the plane. What is the μ-measure of those lines g which cut all of the segments? Let G denote the set of lines in question. Sylvester's work shows that the solution may be expressed as a linear combination

$$\mu[G] = \sum k_j d_j, \qquad j = 1, 2, \ldots, \binom{2n}{2}, \tag{2}$$

where each k_j is an integer, and d_j ranges over all possible euclidean distances between pairs of endpoints of the various segments.

If the segments are in general position so that no three endpoints are collinear, then the integers k_j in (2) remain invariant under perturbation. However, as Ambartzumian stresses, there is a much broader invariance principle hidden in formula (2).

Instead of the usual measure on the cylinder $r = 1$, one may use an arbitrary Borel measure μ which is symmetric with respect to $(0, 0, 0)$. Then, as was recognized by H. Busemann some time ago [2], the right side of (1) defines a distance function d, and very mild restrictions on μ assure that d is a continuous pseudometric on R^2 for which the ordinary lines are geodesics in the sense of shortest paths. Ambartzumian shows that Sylvester's problems have the same solutions in this more general situation. The k_j are unchanged, assuming the "needles" do not vary, and the d_j are defined by formula (1).

The first six chapters of Ambartzumian's book are devoted to the analysis of Sylvester-type problems in E^m. The author's methods are very successful when applied to the classical case $m = 2$, and there is a detailed discussion of techniques for calculating the invariant integers in formulas such as (2). When $m > 2$, suitable extensions of Sylvester's results are much harder to formulate, even though the notion of measure on sets of hyperplanes is easily understood.

For $m = 2$, it was discovered by Ambartzumian (and the reviewer [1], independently, in another context) that formula (1) essentially characterizes metrics on any 2-dimensional geometry for which the lines are geodesics. Each metric is associated with a unique measure on the linesets via formula (1). If the metric happens to be Riemannian, the author calls a geometry a *discoid*. The geodesic measure for a discoid may also be obtained by use of variational methods, but it should certainly be of interest that the existence of this measure is a corollary of a much more general theorem whose proof uses very different techniques.

The discoid offers a rich selection of problems, and Ambartzumian does some beautiful work with isoperimetric inequalities. If A is a subset of a discoid, there is a following formula for the area of A:

$$\pi(\text{Area } A) = \int l(g \cap A)\, d\mu(g), \tag{3}$$

where l is 1-measure along g. Broadly extending an ingenious trick of Pleijel using the measure $\mu \times \mu$, Ambartzumian shows that for any discoid D there is a corresponding nonnegative geodesic function h such that

$$\pi(\text{Area } D) + \int h(g)\, d\mu(g) \leq \frac{1}{2}(\text{Per } D)^2. \tag{4}$$

The inequality $\pi(\text{Area } D) \leq \frac{1}{2}(\text{Per } D)$ follows at once; the open hemisphere is a limiting case here. If D lies in E^2, it turns out that $h(g) = l(g \cap D)$, and thus formulas (3) and (4) at once give the classical isoperimetric inequality. This gives a sample of the pretty applications found in Chapters 7 and 8.

Unfortunately, the most general theory is beset with numerous problems for dimensions exceeding 2. Nonetheless, the author does obtain formulas in the case of the motion invariant measure. This work may be regarded as a hefty extension of the ideas related to Problem 17 in Part IX of Pólya's and Szegö's *Problems and theorems in analysis* **[5]**.

In R^3 the typical metric for which the ordinary lines are geodesics cannot be represented by a Cauchy-Crofton type integral over a measure on the plane-sets. A. V. Pogorelov's recent book **[4]** discusses some aspects of this difficulty. We remark that a Banach space theorem of H. Witsenhausen **[8]** can be combined with variational calculus to show that the integral representation exists if and only if the metric d is of *negative type*. This ties the present work to another active area of metric geometry. For example, L. Dor **[3]** has determined which $L_p^{(m)}$-spaces are of negative type.

In order to obtain analogues of Sylvester's diophantine relation (2) for R^m, $m \geq 2$, the notion of linearly additive metric has been generalized by A. Baddeley to an interesting class of functions defined on complexes. This work is described in Appendix A of the present book. The use of Euler's formula is very neat.

In the final two chapters of the text, which appear to be in a somewhat different spirit from the earlier chapters, Ambartzumian treats several fairly specific topics in mathematical stereology. In a typical stereological problem one has knowledge along certain random lower-dimensional slices of a large collection of m-bodies in E^m, a geometric "poll". Then, subject to an assumed statistical distribution, conclusions are drawn about the average geometric properties of the bodies. Biology has been traditionally a rich source of challenging problems in stereology, and it is likely that the discrete methods of Ambartzumian will be useful in the study of tomographic algorithms, as well as giving further insight into some classical problems.

One hates to find any fault with this unpretentious and highly original work. But as was remarked, the chapters have a rough chronological order with respect to the author's work, and this results in some peculiar positioning of the material. In our opinion the immediate leap, taken in Chapter 1, into the

3-dimensional case of invariant measures on planesets is not the best approach. Surely this topic is more naturally included with those of Chapter 5. How much better to begin with some interesting examples based on the work of Sylvester, Pólya-Szegö, and Pleijel, thus giving the reader a feel for the nice ideas to be treated in depth further on. Incidentally, of the various minor misprints, the misspelling of the name *Schläfli* seems the most eye-catching.

For the casual library reader who wishes to know more about the nature of Ambartzumian's work, we recommend first reading the author's introduction, and then moving to A. Baddeley's well-written Appendix A, which gives an overview of much of the material in the first six chapters of the text and, in addition, several related topics such as Hilbert's Problem IV. As one reads the text, the papers of Baddeley and K. Piefke referenced in Appendix A will also be of interest.

Ambartzumian has established a base camp in a little-explored area of geometry. From here a number of interesting problems can be seen from a new perspective. With luck a boom town could arise. At the very least this work is a significant contribution to the foundations of integral geometry.

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RALPH ALEXANDER

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Stochasticity and partial order, doubly stochastic maps and unitary mixing, by Peter M. Alberti and Armin Uhlmann, Mathematics and its Applications, Vol. 9, D. Reidel Publishing Company, Dordrecht, Holland, 1982, 123 pp., $28.50. ISBN 0-9277-1350-2

Many interesting theorems in functional analysis have their origin in non-trivial finite dimensional results. The book under review provides an example of such a development. It starts with some classical results in convexity theory which go back to Birkhoff and Rado, and it ends up with theorems on C^*-algebras.