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Schrödinger-type operators with continuous spectra, by M. S. P. Eastham and H. Kalf, Research Notes in Mathematics, Vol. 65, Pitman Advanced Publishing Program, Boston, 1982, 281 pp., $24.95. ISBN 0-2730-8526-3

To a mathematical physicist the title *Schrödinger-type operators with continuous spectra* arouses expectations of a wide variety of topics connected with quantum dynamics, so it is something of a surprise to open this book and find no reference to wave operators or the other paraphernalia of scattering theory. Eastham and Kalf soon make it clear, however, that an accurate subtitle might have been *A monograph on the possibility of eigenvalues embedded in the continuum*. The prospective reader should be aware of this narrower scope.

The spectral theorem assigns a unique spectral family, or projection-valued measure on the real line, to any selfadjoint operator on Hilbert space. Von Neumann **[12]** placed the spectral theorem at the heart of his axiomatic formulation of quantum mechanics, and as a result spectral analysis of Schrödinger operators

$$-\Delta + q(x), \tag{1}$$

defined on appropriate subspaces of $L^2(\mathbf{R}^n)$, is a central part of mathematical physics. There are two useful ways to decompose the spectrum of a selfadjoint operator. One of them divides σ_{d}, consisting of discrete eigenvalues of finite multiplicity, from σ_{e}, the essential spectrum, consisting of everything else. The utility of this arises from Weyl's well-known theorems on the invariance of σ_{e} under compact perturbations or, in the case of Sturm-Liouville operators, changes in boundary conditions **[19]**. The second, more measure-theoretic division distinguishes among σ_{p}, the point spectrum, consisting of eigenvalues whether isolated or not; σ_{ac}, the absolutely continuous spectrum, which is associated with the part of the spectral family orthogonal to the eigenvectors and in a sense absolutely continuous with respect to Lebesgue measure; and σ_{sc}, the singular continuous spectrum. The continuous spectrum σ_{c} is the union of σ_{ac} and σ_{sc}. The definition allows the possibility that σ_{c} and σ_{p} intersect. In one-body physics, or two-body physics after removal of the center of mass, the potential q tends to zero at infinity, and typically

$$\sigma_{\mathrm{e}} = \sigma_{\mathrm{c}} = \sigma_{\mathrm{ac}} = [0, \infty), \tag{2}$$

$$\sigma_{\mathrm{sc}} = \varnothing, \tag{3}$$

$$\sigma_p \cap \sigma_c = \varnothing \text{ (or, more rarely, } \{0\}). \tag{4}$$

This, at least, is expected on physical grounds, and there is a significant industry manufacturing theorems guaranteeing (2)–(4), as well as examples where they are false, contrary to naive expectation. This book is concerned with (4). The two ways of dividing up the spectrum go back to the early years of the century, but it took decades for their significance to be widely appreciated. For instance, the term "essential spectrum" was not actually coined until the late forties, by Wintner [**20**]. In a nutshell, the relationship to quantum physics is that σ_p comprises the energies of bound states, and the spectral subspace associated with σ_{ac} consists of dynamical states that may participate in scattering. Usually, it is proved or at least hoped that σ_{sc} simply is not there. However, examples of operators like (1) are known having singular continuous spectra, or at least nowhere dense spectra, or, for that matter, such other amusing features as dense point spectra (for example, see [**2**, **4**, **8**, **11**, **15**]). Some of these are even physically interesting.

If the potential goes to zero at infinity and the energy of a particle is positive, one might expect that quantum fluctuations would eventually propel the particle to a place where its motion would not be confined, so there should not be positive point spectrum. Yet in quantum mechanics there is a less obvious countervailing possibility that some coherence property residing in the potential might confine a particle. It was no effortless matter to make these thoughts precise and understand their implications for quantum mechanics. The great early success of Schrödinger's theory was his revival of Rayleigh's perturbation scheme to calculate the effect of a weak electric field on the energy eigenvalues of a hydrogen atom [**17**], since the old quantum theory could not do this unambiguously. The remarkably good agreement with the experiment known as the Stark effect required some explanation by later scientists who knew that Schrödinger's series diverged and that the spectrum of the Stark-effect operator with nonzero field is purely continuous, with no eigenvalues at all [**3**, **9**]. Actually, Schrödinger was aware of the possibility of continuous spectrum and that its presence near eigenvalues would vitiate his perturbation scheme, but he convinced himself that an innocuous change of variables allowed him to forget about it. Shortly afterwards Oppenheimer wrote his Göttingen dissertation applying the Hellinger-Weyl theory of continuous spectra to quantum mechanics and went on to make a muddled calculation of the ionization rate of hydrogen in an electric field, relying on the assumption that the problem Schrödinger analyzed had no eigenvalues in the continuum [**13**, **14**]. He paraphrased Weyl out of context to justify this [**14**]. Largely in response to Oppenheimer's article, von Neumann and Wigner constructed two explicit examples of one-dimensional potentials producing eigenvalues greater than $\underline{\lim}\, q(x)$. One of them shared the property of the Stark effect potential of going to $-\infty$ as x went to ∞ in some directions, but the other went to 0 as usual. Its peculiarity was a special oscillation that could be thought of as causing coherent reflection. These examples are generally considered the beginning of the subject described in detail by Eastham and Kalf, although the explicit consideration of whether the positive eigenvalue lay in a continuum waited for [**21**]. The construction of examples of operators of type (1) with positive eigenvalues has continued to be a major part of the industry, so that many of the theorems guaranteeing their absence can be considered best possible, at least in the one or two body situations where q is small at ∞ in some sense. As might be expected, the techniques of inverse scattering theory have helped boost the output of positive eigenvalues in recent years.

How does one prove that there are no positive eigenvalues? Eastham and Kalf's very readable account begins with a heuristic argument focusing on a quantity known in physics as the virial, $x \cdot \nabla q$, important for its occurrence in what are known as virial theorems. In their quantum-mechanical form virial theorems state that when u is an eigenfunction of (1) and mild assumptions are made on q, then the expectation value of the virial is twice that of the kinetic energy, or

$$(u, x \cdot \nabla qu) = 2(u, -\Delta u).$$

Of course, $x \cdot \nabla$ is also in essence the generator of the group of dilatations, which is no accident, since one of the principal ways to prove virial theorems is to exploit dilatations. The book contains a separate chapter on virial theorems, a specialty of Kalf. If, for instance,

$$x \cdot \nabla q + 2q < 2\lambda_0 \quad \text{for all } x, \tag{5}$$

then no eigenvalue of $-\Delta + q$ can exceed λ_0. Variants of bound (5) holding only on exterior regions or in some integrated sense can also be used to prove the absence of positive eigenvalues. Eastham's and Kalf's exploration of this idea for one- or two-body situations is thorough and complete, except for one important new technique, which apparently appeared too late for inclusion, *viz.*, that of Froese, Herbst, and the Hoffmann-Ostenhofs [7], who rely on compactness of certain operators involving q and $x \cdot \nabla q$ rather than local or averaged conditions.

The methods vary somewhat according to three cases of interest: (a) one dimension; (b) one or two particles, i.e., n is usually 3, and q in essence tends to zero at infinity; and (c) many particles. Much effort is devoted in the early chapters to the one-dimensional case, where the rich theory of Sturm-Liouville operators, a specialty of Eastham, has much to offer. This subject is surveyed nicely before Eastham and Kalf proceed to give a wide selection of conditions on q forbidding positive eigenvalues in the one-dimensional case. They are also able to deal with alternative situations, such as when $q \to -\infty$ and σ_c is $(-\infty, \infty)$ or when q is periodic and σ_c consists of bands.

The most successful philosophy for banishing positive eigenvalues of higher-dimensional versions of (1) has been that developed by Kato [**10**], Simon [**18**], Eidus [**5**], Agmon [**1**], and others. With control over q and $x \cdot \nabla q$ it is possible to gauge the growth or decay of eigenfunctions of (1) in terms of the integral over $|x| = t$ of a somewhat complicated expression involving the eigenfunction. The expression satisfies inequalities that force the eigenfunction to vanish outside some region. At this point one can appeal to unique continuation theorems for solutions of elliptic equations, which, in analogy to that for analytic functions, state that a solution cannot vanish on an open set without vanishing everywhere. Eastham and Kalf consider these methods as "localized virial techniques".

They made a reluctant decision to leave out another technique, which looks quite different in detail, but must be closely related at some fundamental level. This is the technique of dilatation analyticity, which replaces the study of (1) with that of an object that has been dilated and complexified (see [**16**]). For sufficiently nice potentials, including many of physical interest, this has the effect of moving the essential spectrum into the complex plane while leaving eigenvalues invariant. Once the eigenvalues are no longer embedded in a continuum they are easier to work with. This technique is useful for studying many-particle problems, something which many, particularly those more closely connected with physics, will feel has been somewhat neglected overall by Eastham and Kalf. Only a few relevant theorems are given for the case of many bodies, such as one for homogeneous potentials. In many-particle quantum mechanics, $n = 3k$ and $q(x_1, \ldots, x_n)$ is of the form

$$\sum_{i=1}^{k} V_i(x_j) + \sum_{i \neq j} V_{ij}(x_i - x_j),$$

where each V depends on a three-dimensional spatial coordinate and may tend to 0 at large values, but q is definitely not small when x_i and x_j get large while their difference does not. Several more complicated phenomena arise when there are many bodies. For example, there is a negative portion of the continuum, and it takes nothing more seductive than a symmetry to persuade eigenvalues to live in the negative continuum. Eastham and Kalf may have felt either that it would take too long to treat this case thoroughly or simply that the available theorems were not yet sufficiently definitive. Indeed, immediately upon the publication of the book, Froese and Herbst opened up new territory in this area by exploiting compactness and the Mourre estimate [6]. (Actually, new territory was even opened up in the one-dimensional case by the Hoffmann-Ostenhofs in a paper to appear in *The Journal of Mathematical Physics*. They exclude positive eigenvalues by, for example, bounding dq/dx from *below*, by $(V - \lambda)x^{-1-\delta}$. Also, they have some examples of positive eigenvalues from potentials that go to 0 at ∞ but do not change sign.)

This book was neither intended nor destined to reach a very wide readership. Yet those who are moved to learn about the subject in more detail than the section in [16] will be well rewarded by Eastham's and Kalf's meticulous detail as well as by their generosity with such things as heuristic explanations and open research problems.

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Lectures from Markov processes to Brownian motion, by Kai Lai Chung, A Series of Comprehensive Studies in Mathematics, Vol. 249, Springer-Verlag, New York, 1982, viii + 239 pp., $34.00. ISBN 0-3879-0618-5

What ultimately constitutes a good mathematics book? It seems to the reviewer that this is a function $f(e, r, c)$ of the variables e = effort needed to comprehend the book, r = reward in the form of valuable understanding gained, and c = cost of the book. Of these, the first two are highly dependent on the reader, and, given the first two, dependence on the third is completely individual, hence need not be discussed here. We assume f is decreasing in e and increasing in r. On these assumptions, Chung's book comes out very well indeed for the present reviewer. But let us beware. The reviewer has recently written a book [*Essentials of Brownian motion and diffusion*, Math. Surveys, vol. 18, Amer. Math. Soc., Providence, R.I., 1981] which complements Chung's book to a considerable degree. It gets one over the hard beginning (especially §1.3, Optional Times) without appreciably encroaching on the content. For the two books together one might suggest the title *From Brownian motion to Markov processes and back*.