13. R. Johnson, *A linear almost periodic equation with an almost automorphic solution*, Proc. Amer. Math. Soc. **82** (1981), 199–205.

14. N. Bogolyubov, *O nekotorykh statisticheskikh methodakh v matematicheskoĭ fizike* (*On some statistical methods in mathematical physics*), Akad. Nauk Ukrainskoĭ SSR, 1945.

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Geometrical methods in the theory of ordinary differential equations, by V. I. Arnold, Grundlehren der mathematischen Wissenschaften, Volume 250, Springer-Verlag, New York, 1983, x + 334 pp., $36.00. ISBN 0-3879-0681-9

> Among all mathematical disciplines the theory of differential equations is the most important.
> S. Lie (1895)

> [The work of] Smale... shows that the problem of the complete topological classification of differential equations with high dimensional phase space is hopeless...
> V. Arnold (p. 87)

> One picture is worth a thousand symbols.
> Old proverb

Poincaré drew an analogy between algebraic and differential equations. In solving an algebraic equation one first does a *qualitative* investigation, determining the number of real roots by Sturm's theorem; then one carries out the *quantitative* step of numerically evaluating the roots. Similarly with the study of algebraic curves: only after the qualitative step of determining which branches are closed or infinite does one numerically find a certain number of points on the curve. It is the same with differential equations: before numerically evaluating the solution, first one should perform a qualitative investigation into the general form of the solution. Is it bounded or unbounded? Does it oscillate, or converge, or neither? Is it stable or unstable? This last question involves looking at not just a single solution, but all the solutions. In connection with this, Hadamard suggests another parallel with algebraic equations: great progress was made only after Galois and others began to look at the relations between *all* the roots of a polynomial.

The essence of Poincaré's "qualitative" investigations, according to Hadamard, is to regard the values of the unknown function not as a function of the independent variable (usually interpreted as time), but rather as a function of the initial data. The more recent notion of "dynamical system" is an abstract formulation of this point of view.

The initial data (of an autonomous ordinary differential equation) lie in the space of all possible initial data, called the phase or state space, usually a differentiable manifold. The solutions to the equation form a system of smooth parametrized curves, one through each point of the phase space. For each real number t we obtain, by following these curves, a map from the phase space into itself. Thus we are in the geometric realm of spaces and maps, or topology. Qualitative methods are inherently geometric in character.

Arnold's book is an introductory survey, by one of the masters of the subject, to some of the most important recent work in geometrically oriented dynamical systems theory.

What are "geometrical methods"? To attain some perspective on this question let us go back to the founder of Arnold's subject. In a provocative essay in *La valeur de la science*, Poincaré averred that "one is born a mathematician, one doesn't become one, and it seems also that one is born either a geometer or an analyst." Neither subject matter nor education is the determining factor, but "*l'esprit*". Analysts, concerned with logic and rigor, proceed carefully step by step; geometers, guided by intuition, make rapid but precarious progress. Hermite, Weierstrass, Kowalewski and Euclid were analysts; Bertrand, Riemann, Klein and Lie were geometers. Both are necessary, but analysts are less common. The analyst seeks his vision from within, *the geometer sees pictures in space*.

This suggests that a geometrical method is a way of thinking about mathematics in which visual intuition and analogy play a predominant part. Its chief role is to supply images which instantly recall the defining relations between the main concepts, *and which portray new relations*—thus we can literally *see* a theorem.

Consider for example "a tangent vector to the unit sphere in n-space". Do you see a ball with a little arrow lying on it? Then (for the moment) you are a geometer. You are an analyst if you prefer instead to think about "a one-jet of a differentiable function from the real numbers into the set $\{x \in \mathbf{R}^n: |x| = 1\}$". Now think about a vector field on the 2-sphere: the geometer sees the ball covered with arrows, their directions varying continuously. Actually the geometer has trouble with this picture—one can't quite imagine the *whole* ball covered with arrows. There must be a spot where... Aha! Theorem. *Every vector field on the 2-sphere has a singularity.* (If you are a *very* good geometer—if you are Poincaré—you will see similar pictures for other surfaces, and if you are also an analyst you may assign numbers to the singularities and relate the sum to the genus of the surface.)

The importance of geometrical thinking—and the existence of other kinds of thinking—was brought home to me as a graduate student. I complained to a fellow student in a course on algebra that I disliked the subject because there were no pictures I could associate with the mathematics. "But that's just what I like about this class," I was told. "I can never understand those darn pictures the topologists are always drawing." It was astonishing to learn that there were other ways of thinking about mathematics. Upon being asked what went through his mind when he studied algebra, my friend said something like, "Oh, we want to push this term to the other side of that one, but then we need to add a correction term over here...."

My colleague Murray Protter once said that a topologist accepts a proof if and only if when reading it, he sees the same picture that the author saw when writing it. I think this also applies to geometric methods in other fields.

A geometrical method is one that uses the language and methods of geometry, including that of topology: manifold, fibre bundle, transversality, etc. Topology is the study of maps between spaces. When the two spaces are the same there is the possibility of *iteration* of the map. Analysis is largely about iteration of a process with emphasis on the limit of the process—and limits are also in the domain of topology. A geometric method in analysis creates *new spaces* in which the process can be interpreted—seen!—as maps.

Consider the following example, whose elaboration is at the heart of much of Arnold's book. A vector field on a manifold M is, in geometrical language, a cross-section of the tangent vector bundle TM. (It is hard to remember that there is also an older, less geometrical definition: a contravariant tensor of order one.) A singularity of the field means a point where the value of the cross-section is zero (or all the components of the tensor vanish). The singularity is *simple* if it occurs at a place where the cross section is transverse to the set of zero vectors (or in local coordinates, where the Jacobian determinant is nonzero). It is a basic theorem that every vector field can be approximated, in any reasonable sense, by one having only simple singularities. This result becomes totally obvious if one sees the standard picture of the cross section: TM is the plane, the zero set is the horizontal axis, the cross section is a curve which is the graph of a function, a singularity is a point where the curve meets the horizontal axis. To make all singularities simple, raise the graph slightly so that wherever it meets the horizontal axis it crosses it at a nonzero angle.

This of course is not a proof, but it is a powerfully convincing argument. The proof requires a form of the Brown-Dubovicki-Morse-Sard theorem. But for one who knows that theorem it is easy to produce a proof—*if* one sees vector fields as cross-sections.

Topological language is indispensable for global problems in analysis (and in fact topology came directly from such problems). But it is also very useful in local questions.

Arnold's book is mainly about local questions, although some topics deal with differential equations on n-dimensional tori and other manifolds. But global considerations enter in another way. Just about everything in the book has to do, directly or indirectly, not with individual equations but with large families of equations. Thus one of the important spaces is the function space S of all appropriate vector fields (or diffeomorphisms) on a manifold, under a suitable topology. Certain of these vector fields are considered degenerate in some way—for example those that have a nonsimple singular point. These form a subset $V \subset S$. The intrinsic geometry of V and its extrinsic geometry in S are the key to many investigations. Suppose for instance that V consists of all vectors fields having a nonsimple zero. The statement that V is nowhere dense in S is just a rephrasing of the approximation theorem stated above.

Suppose we are interested not in single vector fields, but in k-parameter families of fields. It is natural to represent such a family by a continuous map $F\colon \mathbf{R}^k \to S$. Can we approximate the whole map F so that its image avoids the set V of degenerate fields? Not if $k > 0$, because it can be shown that V has codimension one in S. For various values of k, what can we say about the fields represented by points in $F^{-1}(V)$ for a "generic" map F? For which values of k can we perturb F so that none of the degenerate fields have zeros making second-order contact with the zero set? These questions can be attacked by forming further subspaces of V which represent various other kinds of degeneracies, and studying their geometry together with the geometry of their inverse images under F.

All this is explained clearly in the introduction to Chapter 6, followed by the development of the necessary technical tools—jet spaces, transversality, stratified varieties, Newton polygons, etc. These are then applied to several topics: matrix families, bifurcations of singular points of vector fields, loss of stability of equilibria and self-sustained oscillations, and others. One of the most interesting discussions concerns the "construction of an elliptic curve [over the complex numbers] from a resonant invariant manifold". The complex geometry of these curves and their holomorphic normal bundles yields invariants which are applied to the problem of linearizing a local diffeomorphism of $\mathbf{C}^2$ by an analytic change of variables—a powerful use of geometry, and a surprising one (to me) since the linearization problem is purely local, yet an elliptic curve is a global object.

I have discussed this last chapter first because it gives a clear instance of what the author means by geometrical methods, and because I think it is the high point of the book. Not only are the explanations very clear, even though (or perhaps because) many things are only sketched, but much of this important material is available only in the Russian literature.

The contents and point of view of the book are well described by the author in his preface, from which I quote:

> In the selection of material for this book the author intended to expound basic methods applicable to the study of differential equations. Special efforts were made to keep the basic ideas (which are, as a rule, simple and intuitive) free from technical details. The most fundamental and simple questions are considered in the greatest detail, whereas the exposition of the more special and difficult parts has been given the character of a survey.
>
> The book begins with the study of some special differential equations integrable by quadrature....
>
> The theory of partial differential equations of the first order is considered by means of the natural contact structure in the manifold of 1-jets of functions. The necessary elements of the geometry of contact structures are developed....
>
> A significant portion of the book is concerned with methods which are usually called *qualitative*.... The book discusses the analysis of differential equations from the point of view of structural stability, that is, the stability of the qualitative picture with respect to a small change in the differential equation....

> The most powerful and frequently applicable methods of study of differential equations are the various asymptotic methods. We develop the basic ideas of the averaging method going back to the work of the founders of celestial mechanics and widely usable in all those areas of applications, where a slow evolution has to be separated from fast oscillations....
>
> ... In this book we describe the main results of the method of Poincaré normal forms, including a proof of Siegel's fundamental theorem on the linearization of a holomorphic mappping....
>
> This book concludes with a chapter on bifurcation theory, in which the methods developed in the preceding chapters are applied, and the main results obtained in this field, beginning with the fundamental work of Poincaré and Andronov, are described.
>
> In discussing all these subjects, the author attempts to avoid the axiomatic-deductive style, with its unmotivated definitions concealing the fundamental ideas and methods; similar to parables, they are explained only to the disciples in private.
>
> ... The author attempts to write in such a way that this book can be read not only by mathematicians, but also all users of the theory of differential equations.
>
> We only assume a little general mathematical knowledge on the part of the reader... for example familiarity with the textbook V. I. Arnold, *Ordinary Differential Equations*... is sufficient (but not necessary).

In this as in his other writings, Arnold exhibits a thorough mastery of the material and great gifts for exposition. These have combined with his strongly expressed opinions to make a most informative, useful and stimulating book, from which I learned a lot. It is not, however, without flaws.

The most exasperating fault, inexcusable in the computer age, is that *there is no index*! Nor is there a bibliography—references are scattered throughout the book. This is made worse by the author's policy to "avoid references from one chapter to another, and even from one paragraph to another."

There are few misprints, but the translation has occasional awkwardnesses, e.g. "continuous fractions", "Tarsky", and a reference on p. 330 to a book on matrices, apparently published solely in Moscow, by one "F. R. Gantmaher".

There is a substantive weakness as well. The author's laudable intention of emphasizing ideas rather than technicalities requires that many proofs and even definitions be merely sketched. In many places this is done with great skill; but in others it results in such vagueness that the nonexpert reader will probably not understand, and may even be misled. I found several examples in the last three sections of Chapter 3, devoted to hyperbolic theory (the topic in this book that I know best).

These difficulties are curiously related: they all have to do with the stable and unstable manifolds of hyperbolic fixed points and of Anosov diffeomorphisms—the very heart of hyperbolic theory. For example, the invariant manifolds associated to a hyperbolic fixed point are never defined—they are only referred to, in small type on p. 128, as a "special case" of the construction of the expanding and contracting invariant foliations of Anosov diffeomorphisms. The definition of "foliation" is given, in a footnote on p. 127, only for smooth foliations; but those associated to an Anosov diffeomorphism are not necessarily smooth. Just after this footnote it is pointed out that the fields of tangent planes to the foliations are not smooth.

Another difficulty is in the application of the Hartman-Grobman theorem theorem (p. 141) to prove that unstable manifolds of a fixed point vary C^1 continuously with the diffeomorphism. Since the theorem yields only a *topological* conjugacy between a hyperbolic local diffeomorphism and its linear part, it is hard to see how it justifies C^1 continuity.

A further confusion about stable and unstable manifolds occurs on p. 175 in the section on averaging. An unexplained picture shows (presumably) two fixed points, with the stable manifold of one crossing several times the unstable manifold of the other. This configuration is what Poincaré and all later writers have called *heteroclinic*, the term *homoclinic* being reserved for the case where the fixed points coincide. Yet the reference to the diagram refers to a "homoclinic picture". This is followed by a footnote defining homoclinic point, adding to the confusion by requiring, against standard (and Poincaré's) usage, that the stable and unstable manifolds be distinct.

Great importance attaches to homoclinic points and their orbits. Poincaré discovered them and showed in certain cases (now known to be generic), that where there is one there are infinitely many having totally different dynamic behavior. This was the first inkling of the chaos lurking behind simple-looking differential equations. Homoclinic orbits were further studied by Birkhoff, who showed that (in some situations) they are limits of periodic orbits of unbounded periods. Smale showed they exist in structurally stable situations (his famous "horseshoe" system), and more recently they have been found in many natural dynamical systems. They are the best understood source of chaotic dynamics. While that is not one of the main topics of this book, it is alluded to. A more careful treatment would be desirable.

There are other difficulties of a similar nature. It may be, however, that the nonexpert reader would not in fact be bothered by them! For I must admit that I found few in the sections on topics that I am less familiar with.

In any case these flaws are minor compared to the great virtues of the book: It is a very illuminating and highly readable exposition of interesting topics, which are of great relevance both to theory and applications. It has an even rarer virtue: Arnold includes many discussions of the history and significance of the mathematics, including its relationship to physics and experimental observation and the relative usefulness of different approaches. His own point of view is clearly expressed. Some of his judgments are controversial, which will have the good result of stimulating discussion. These digressions provide a larger intellectual framework for the mathematics, adding coherence and meaning to the abstract theory. For the reader who does not know the historical and scientific background, this is very valuable: it provides motivation and a way of thinking *about* the mathematics.

A most informative, stimulating, and refreshing book!

MORRIS W. HIRSCH