BOOK REVIEWS

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Almost periodic functions and differential equations, by B. M. Levitan and V. V. Zhikov, Cambridge University Press, New York, 1983, x + 211 pp., $34.50. ISBN 0-5212-4407-2

The problem of finding almost periodic solutions to differential equations has a variety of themes. An initial one is called the Bohr-Neugebauer Theory. The elementary result, due to Harold Bohr, is that the indefinite integral of an almost periodic function is almost periodic if and only if it is bounded. Bohr and Neugebauer [1] extended this to arbitrary order constant coefficient equations, and thus to vector systems. The most general theorem along this line is offered by Bochner [2], who studies differential-difference equations.

After the discovery of Bochner [3] of a new definition of almost periodic function it was possible to look at almost periodic solutions to differential equations in the context of topological dynamics. Initially, this application of topological dynamics was restricted to autonomous equations or the Poincaré map of a periodic system. However in 1965, Miller [4] and Millionshchikov [5] showed how nonautonomous equations could be embedded in dynamical systems. A good exposition is given by Sell [6].

A third theme has its origins in a paper of Favard [7]. He considered the equation

$$x' = A(t)x + f(t) \tag{1}$$

with x a vector, f an almost periodic vector and A an almost periodic matrix. If the homogeneous equation has all its bounded solutions x satisfying $\inf_t |x(t)| > 0$, then the convex set of solutions to the nonhomogeneous equation has a unique solution nearest to the 0 function. With an additional technical hypothesis, the hull hypothesis, this solution is almost periodic. The mini-max approach is widely used. Generally, if any one of a class of functionals is uniquely minimized by a solution of an almost periodic equation, then that minimizing solution is almost periodic. The book of Amerio and Prouse [8] is devoted to developing one version of this idea for energy functionals for partial differential equations.

In 1955 Amerio [9] introduced the notion of separated solutions and showed how this led to the existence of almost periodic solutions. Many investigators, including Seifert, Miller, and Yoshizawa, realized that this idea was very useful in showing how stability conditions are connected with almost periodic solutions. A solution $x(t)$ is separated in a compact set K if $x(t) \in K$ for each t and there is a constant ρ such that if y is another solution with $y(t) \in K$ for all t, then $|x(t) - y(t)| \geqslant \rho$. If x is a uniformly stable solution in K then all other solutions that start in K near to x remain near x in the future. But going backward in time the solutions that remain in K stay apart if x is to be uniformly stable. For autonomous equations stable solutions are also uniformly stable. Thus stable solutions tend to be separated. A major breakthrough in use of these methods is represented by Bochner's **[2]** discovery of a pointwise definition of almost periodic function. Duffing and Lienard type equations seem well suited to this type of analysis. The connections between stability and almost periodicity are exposed in Yoshizawa **[10]**.

The direct construction of almost periodic functions for quasi-periodic equations is most common among investigators in classical mechanics. The construction of formal series solutions is not difficult, but proving the appropriate convergence is another matter. The major difficulty is called the small divisor problem. Solutions to the small divisor problems were given independently by Arnold, Kolmogorov and Moser. For an exposition of these results, see Moser **[11]**. More recent developments along these lines use sophisticated implicit function theorems fashioned along the arguments of Moser's book.

Meanwhile, the linear equation (1) continues to be the subject of research. To see why this should be, recall the Floquet theory when A is a periodic matrix. There exists an invertible periodic matrix $P(t)$ so that the change of variable $y = Px$ carries (1) into $y' = By + g$ where B is a constant matrix and g has the same qualitative properties as f. When A is almost periodic and not periodic, then it would be natural to search for a similar change of variable with P almost periodic. Unfortunately, the structure of the solution space of the homogeneous equation $x' = Ax$ is much more complicated in the almost periodic case and B cannot in general be selected to be a constant. The notions of admissibility and dichotomy are closely related to efforts to overcome this lack of a Floquet theory. Equation (1) is admissible for a pair of Banach spaces (X, Y) if there is a solution in X for every $f \in Y$. The homogeneous equation $x' = Ax$ satisfies an exponential dichotomy in X if there exists a projection in P on X so that for the fundamental solution Φ, estimates of the form

$$\left|\Phi(t)P\Phi(s)^{-1}\right| \leqslant Ke^{-\alpha(t-s)}, \qquad t \geqslant s,$$
$$\left|\Phi(t)(I-P)\Phi(s)^{-1}\right| \leqslant Ke^{-\alpha(t-s)}, \qquad t \leqslant s,$$

hold for certain positive constants K and σ. This essentially says that the initial conditions split into subspaces on which exponential decay (or growth) occurs either forward or backward in time. In some contexts P may depend on t. In any case, the existence of dichotomies is equivalent to admissibility, while dichotomies are more directly related to stability. For questions of dichotomies, the papers of Sacker and Sell (see **[12]** as an example) deal with a more general topic, but almost periodic systems are an important special case. Further research along this line is being done by Johnson; see [13] as an example.

Perturbation methods are useful in the context of strong stability results. This is commonly understood in general settings. The almost periodic case offers a special form which leads to different kinds of results. Suppose one is considering an equation of the form $x' = f(x, t)$ where $f(x, t)$ is an almost periodic function. An idea developed by Bogolyubov [14] is to consider this equation as a perturbation of $x' = f_0(x)$, the so called averaged equation, with

$$f_0(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x, t)\, dt.$$

If this autonomous equation has a stationary point or almost periodic solution which is stable, one expects to find almost periodic solutions of the original equation nearby.

The book under review is an exposition of the work of Zhikov on almost periodic solutions. The first five chapters, written by Levitan, develop the theory of almost periodic functions with values in a metric space or a Banach space. A chapter is devoted to the theory of weak almost periodic functions. As such, it parallels the first few chapters of [4]. The exposition is quite detailed.

The second part of the book was written by Zhikov. Chapter 6 deals with the Bohr-Neugebauer theme and the harmonic analysis of bounded solutions. Next, Zhikov outlines some of the results connecting the ideas of separating solutions and stability. The novel feature of the seventh chapter here is that Amerio's paper [9] is cast into the context of dynamic systems. The separated condition is replaced by the distal flow hypothesis. Chapter 8 exposes Favard theory in Banach space both for almost periodic and weakly almost periodic solutions. The exposition is an amalgam of Favard's and Amerio's ideas.

The rudiments of variational inequalities are discussed in Chaper 9. This theory is applied to finding almost periodic solutions of the evolution equation. Chapter 10 considers the questions of admissibility and dichotomy for linear equations in a Banach space. Most of this material seems to be standard. Finally Chapter 11 addresses the method of averaging for parabolic partial differential equations.

The exposition in the book is uneven, reflecting the dual authorship. The first part, written by Levitan, is detailed and easily readible, the second part varies. The proofs are most sketchy where they make use of results from topology or functional analysis. Many theorems have short statements. This is possible because the hypotheses are not explicitly given. Some hypotheses are contained in a discussion previous to the statement of the theorem, and it may require close reading of several pages in order to find a complete statement of the theorem.

The translation seems to generally be a very careful one. Minor slips occur. The same notion appears as f-increasing at one place and later as f-returning. What is called semicontinuous in this book is usually hemicontinuous.

The bibliography refers mostly to the Soviet literature in the subject. There are no references to much of the literature I have cited above. An interested reader may want to use them to supplement the literature in the book to gain some insight into the Western literature. In part, the authors do not seem to care about citations. On p. 123, for example, a cited work of Ellis (1958) is supposed to have used the work of Furstenberg (1961). Miller and Sell were supposed to have developed the idea of the dynamical system approach to nonautonomous equations later than Millionshchikov. In fact [**4** and **5**] appeared in the same year. Having said that, it is curious that no citation to Miller is ever given and one to Sell has apparently been added by the translator, and this one is not a dynamical systems paper.

Levitan has defined N-almost periodic functions. They are briefly mentioned in this monograph. This notion is essentially equivalent to almost automorphy introduced by Bochner in [**2**]. Almost automorphy is a natural condition in both the Favard and Bohr-Neugebauer theories. The authors mention [**2**] but do not take the opportunity to harvest the theorems about almost automorphic solutions when they would follow easily from the given discussions. Moreover, they do not seem to be aware of the power of Bochner's pointwise definition.

This book is not a comprehensive account of the theory of almost periodic differential equations. Indeed, with a published literature that includes more than 700 papers, it is not likely that a comprehensive book will appear. I would characterize the present volume as a set of lecture notes on Zhikov's special interests. Making his results available in a logically coherent manner in English is useful to the scientific community.

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Geometrical methods in the theory of ordinary differential equations, by V. I. Arnold, Grundlehren der mathematischen Wissenschaften, Volume 250, Springer-Verlag, New York, 1983, x + 334 pp., $36.00. ISBN 0-3879-0681-9

> Among all mathematical disciplines the theory of differential equations is the most important.
> S. Lie (1895)

> [The work of] Smale ... shows that the problem of the complete topological classification of differential equations with high dimensional phase space is hopeless ...
> V. Arnold (p. 87)

> One picture is worth a thousand symbols.
> Old proverb

Poincaré drew an analogy between algebraic and differential equations. In solving an algebraic equation one first does a *qualitative* investigation, determining the number of real roots by Sturm's theorem; then one carries out the *quantitative* step of numerically evaluating the roots. Similarly with the study of algebraic curves: only after the qualitative step of determining which branches are closed or infinite does one numerically find a certain number of points on the curve. It is the same with differential equations: before numerically evaluating the solution, first one should perform a qualitative investigation into the general form of the solution. Is it bounded or unbounded? Does it oscillate, or converge, or neither? Is it stable or unstable? This last question involves looking at not just a single solution, but all the solutions. In connection with this, Hadamard suggests another parallel with algebraic equations: great progress was made only after Galois and others began to look at the relations between *all* the roots of a polynomial.

The essence of Poincaré's "qualitative" investigations, according to Hadamard, is to regard the values of the unknown function not as a function of the independent variable (usually interpreted as time), but rather as a function of the initial data. The more recent notion of "dynamical system" is an abstract formulation of this point of view.