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ON THE YANG-MILLS-HIGGS EQUATIONS

BY CLIFFORD HENRY TAUBES[1]

I. Introduction. The purpose of this note is to announce new results for the Yang-Mills-Higgs equations on R^3. These SU(2) Yang-Mills-Higgs equations are a set of partial differential equations where the unknown is a pair, $c = (A, \Phi)$, with A a connection on the vector bundle $E = R^3 \times su(2)$ and Φ a section of E. Here $su(2)$ =Lie alg. SU(2). These equations are

$$D_A * F_A + [\Phi, D_A\Phi] = 0, \qquad D_A * D_A\Phi = 0, \tag{1}$$

with boundary condition $\lim_{|x|\to\infty} |\Phi|(x) = 1$. Here, the notation follows [**1**]. That is, F_A is the curvature of A, D_A is the exterior covariant derivative on $\bigwedge T^* \otimes E$ and $[\cdot,\cdot]$ is the natural, graded bracket on $\bigwedge T^* \otimes E$: If ω, η are, respectively, E-valued p, q forms, then $[\omega, \eta] = \omega \wedge \eta - (-1)^{pq}\eta \wedge \omega$. The $*$ in (1) is the Hodge star on $\bigwedge T^*$ from the Euclidean metric on T^*. The norm $|\cdot|$ on $T^* \otimes E$;is that induced from the Euclidean metric on T^* and the Killing metric on $su(2)$.

Equation (1) is the variational equation of an action functional

$$\mathcal{A}(A, \Phi) = \frac{1}{2} \int_{R^3} \{|F_A|^2(x) + |D_A\Phi|^2(x)\}\, d^3x. \tag{2}$$

One is to consider $\mathcal{A}$ as a function on the set

$$\mathcal{C} = \{\text{smooth } (A, \Phi) : \mathcal{A}(A, \Phi) < \infty \text{ and } 1 - |\Phi|(x) \in L^6(R^3)\}. \tag{3}$$

$\mathcal{C}$ is topologized as follows [**2**]: Let θ denote the flat, product connection on E. The topology of $\mathcal{C}$ is defined to be the weakest for which the map sending $C = (A, \Phi) \in \mathcal{C}$ to

$$(A - \theta, \mathcal{A}(c)) \in \Gamma(T^* \otimes E) \times \Gamma(E) \times [0, \infty)$$

is continuous.

The topological group

$$\begin{aligned} \mathcal{G} &= \{\text{smooth, unitary automorphisms of } E\}, \\ &= C^\infty(R^3; \mathrm{SU}(2))/\{\pm 1\} \end{aligned} \tag{4}$$

acts continuously on $\mathcal{C}$ and leaves $\mathcal{A}$ and (1) invariant. The subgroup $\mathcal{G}_0 = \{g \in \mathcal{G} : g(0) = 1\}$ acts freely on $\mathcal{C}$. Let $\mathcal{B} = \mathcal{C}/\mathcal{G}_0$ denote the quotient. The functional $\mathcal{A}$ descends as a continuous, SO(3) invariant function on $\mathcal{B}$.

The relationship between $\mathcal{A}$ and $\mathcal{B}$ is described in the following theorems.

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[1]NSF Postdoctoral Fellow in Mathematics

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THEOREM 1. *The space $\mathcal{B}$ is homotopic to the space of smooth maps from S^2 to S^2,* Maps$(S^2; S^2)$. *The space $\mathcal{B}$ has a compatible Fréchet space structure for which $\mathcal{A}$ is a smooth function. With this smooth structure, a point $[A, \Phi] \in \mathcal{B}$ is a critical point of $\mathcal{A}$ if and only if $(A, \Phi) \in \mathcal{C}$ satisfies* (1).

Thus, $\mathcal{B}$ is topologically the disjoint union of spaces $\mathcal{B}_n$, $n \in Z$, with each $\mathcal{B}_n$ homotopic to the space of degree n maps from S^2 to S^2 (cf. [**2**, §3 and Appendix 2]).

It has previously been established that $\mathcal{A}$ takes on its minimum on each $\mathcal{B}_n$ [**3, 4, 5**], and, by a min-max argument, that $\mathcal{A}$ has a nonminimal critical point on $\mathcal{B}_0$ [**2, 6**]. A new min-max theory for $\mathcal{A}$ on $\mathcal{B}$ and the nontrivial topology of $\mathcal{B}_n$ results in

THEOREM 2. *For each n, $\mathcal{A}$, when restricted to $\mathcal{B}_n$, has an unbounded set of critical values.*

Restrict $\mathcal{A}$ to $\mathcal{B}_n$, $n \in Z$. One observes that $\mathcal{A} \geq 4\pi|n|$, with equality at the minima on $\mathcal{B}_n$ where the Bogomol'yni equations are satisfied [**7**]:

$$F_A = \operatorname{sign}(n) * D_A \Phi. \tag{5}$$

The set of $[A, \Phi] \in \mathcal{B}_n$ which solve (5) is called the moduli space, $\mathcal{M}_n$, of self-dual monopoles. One can study these spaces using twistor techniques [**8, 9, 10**]. But still, little is known about them.

By applying the new min-max theory as in [**11**], one can study the topology of $\mathcal{M}_n$ via the embedding of $\mathcal{M}_n$ in $\mathcal{B}_n$. Of import here is an observation from [**12**] that the hessian of $\mathcal{A}$ at a nonminimal critical point in $\mathcal{B}_n$ must have index larger than $|n|$. One obtains as a result

THEOREM 3. *For each n, the inclusion $\mathcal{M}_n \to \mathcal{B}_n$ induces an isomorphism of the pointed homotopy groups $\pi_l(\cdot)$ for $0 \leq l \leq |n| - 1$ and an epimorphism of $\pi_{|n|}(\cdot)$.*

The details and proofs of these results are forthcoming [**13**]. The motivation for the new min-max theory for $\mathcal{A}$ is obtained by considering the following isotopy of $\mathcal{B}$: First, flow from $c = [A, \Phi]$ along a vector field which is minus the gradient of $\mathcal{A}$ on domains in R^3 where $\Psi \equiv (|F_A|, |D_A\Phi|, 1 - |\Phi|)$ is small, and which is zero elsewhere. This flow converges since it is a perturbation of a linear flow. The limit, $c' = [A', \Phi']$ satisfies (1) on domains in R^3 where Ψ is small. Next, flow along minus the gradient flow of $\mathcal{A}$, projected to maintain (1) in the small Ψ domains. The a priori estimates from [**1**] for Ψ in the small Ψ regions (where (1) is satisfied) allow one to prove that the *large* Ψ regions remain in a bounded set on R^3 for all $t \in (0, \infty)$. Under these conditions, K. Uhlenbeck's compactness theorems [**14**] imply convergence for the flow.

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Department of Mathematics, University of California, Berkeley, California 94720