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A p-ADIC REGULATOR PROBLEM IN ALGEBRAIC K-THEORY AND GROUP COHOMOLOGY

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Let $\mathcal{O}$ be the ring of integers in a number field F. Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal and $\mathcal{O}_{\mathfrak{p}} = \varprojlim \mathcal{O}/\mathfrak{p}^s$ be the $\mathfrak{p}$-adic completion of $\mathcal{O}$. Let

$$\hat{K}_n(\mathcal{O}) = K_n(\mathcal{O}) \quad \text{mod torsion},$$
$$\hat{K}^c_n(\mathcal{O}_{\mathfrak{p}}) = K^c_n(\mathcal{O}_{\mathfrak{p}}) \quad \text{mod torsion},$$

where $K_n(\mathcal{O})$ is the algebraic K-theory of Quillen [**Q**] and

$$K^c_n(\mathcal{O}_{\mathfrak{p}}) = \varprojlim K_n(\mathcal{O}/\mathfrak{p}^s)$$

is the "continuous" or "p-adic" algebraic K-theory of $\mathcal{O}_{\mathfrak{p}}$ studied in [**W1**] by Milgram and the author. Results of [**B**] and [**W1**] suggested asking whether

$$\Phi_{\mathfrak{p}}\colon \hat{K}_n(\mathcal{O}) \to \hat{K}^c_n(\mathcal{O}_{\mathfrak{p}}) \tag{1}$$

or

$$\Phi\colon \hat{K}_n(\mathcal{O}) \to \bigoplus_{\mathfrak{p}|p} \hat{K}^c_n(\mathcal{O}_{\mathfrak{p}}) \tag{2}$$

is injective, where p is a fixed rational prime and $n > 1$ is odd. Observe that each $\Phi_{\mathfrak{p}}$ is clearly injective for $n = 1$, because $K_1(\mathcal{O}) = \mathcal{O}^*$ and $K^c_1(\mathcal{O}_{\mathfrak{p}}) = \mathcal{O}^*_{\mathfrak{p}}$. A much harder problem is whether $\Phi \otimes \mathbf{Z}_p$ is injective. For $n = 1$ and F totally real abelian, injectively of $\Phi \otimes \mathbf{Z}_p$ on the subgroup of $\mathcal{O}^*$ consisting of those elements congruent to $1 \bmod \mathfrak{p}$ for each $\mathfrak{p} \mid p$ is equivalent to nonvanishing of the p-adic regulator [**Br**, **C**]. As an example of (1) let F be quadratic imaginary. Then is

$$\Phi_{\mathfrak{p}}\colon \mathbf{Z} \cong \hat{K}_3(\mathcal{O}) \to \hat{K}^c_3(\mathcal{O}_{\mathfrak{p}}) \cong \mathbf{Z}_p \tag{3}$$

injective when $p = \operatorname{char}(\mathcal{O}/\mathfrak{p})$ is unramified with $\mathcal{O}_{\mathfrak{p}} \cong \mathbf{Z}_p$? J.-P. Serre asked an equivalent cohomological version of (1) and (2) prior to the circa 1975 K-theory formulation. For special case (3) injectivity is equivalent to showing $\Phi_{\mathfrak{p}} \otimes Q_p$ is an isomorphism, which in turn amounts to showing

$$Q_p \cong H^3_c(\mathrm{SL}_n(\mathcal{O}_{\mathfrak{p}}); Q_p) \to H^3(\mathrm{SL}_n(\mathcal{O}); Q_p) \cong Q_p \tag{4}$$

is an isomorphism for n large. H^3_c denotes the continuous cohomology of the p-adic group $\mathrm{SL}_n(\mathcal{O}_{\mathfrak{p}})$ and H^3 is the Eilenberg-Mac Lane cohomology of the discrete group $\mathrm{SL}_n(\mathcal{O})$. Compare [**L**]. Numerous examples of (4)

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result from nonvanishing of the Gross-Coleman Q_p-regulator as formulated in [**Co**]. This regulator connects the p-adic dilogarithm and the L-function values $L_p(2, \chi\omega^{-1})$.

There is the companion $\mathbf{Z}_p$-regulator question to (3): namely, determine the index R_p of

$$(5) \qquad \Phi_{\mathfrak{p}} \otimes \mathbf{Z}_p \colon \mathbf{Z}_p \cong \hat{K}_3(\mathcal{O}) \otimes \mathbf{Z}_p \to \hat{K}_3^c(\mathcal{O}_{\mathfrak{p}}) \cong \mathbf{Z}_p.$$

For $F = Q(\sqrt{-3}) = Q(\mu)$, where $\mu^3 = 1$, we give examples for which $R_p = 1$, i.e., for which $\Phi_{\mathfrak{p}} \otimes \mathbf{Z}_p$ is an isomorphism. This is done with the aid of a homomorphism $\mathrm{Ch}_{\mathfrak{p}} : K_3(\mathcal{O}) \to \mathbf{Z}/p$ constructed by elementary methods, and the values of $\mathrm{Ch}_{\mathfrak{p}}$ turn out to be related experimentally to the values $L_p(2, \chi\omega^{-1}) \bmod p$, where χ is the Dirichlet character of conductor 3 and ω is the Teichmüller character on $\mathbf{Z}_p^*$. The details are in [**W2**].

Aisbett [**A**] has shown $K_3(\mathbf{Z}/p^n) = \mathbf{Z}/p^{2(n-1)} \oplus \mathbf{Z}/p^2 - 1$ for $p > 2$. To test for examples where (5) is an isomorphism it is sufficient to

(a) find an explicit element $B \in K_3(\mathcal{O})$,

(b) find an explicit formula for a homomorphism

$$\mathrm{Ch}_{\mathfrak{p}} : K_3(\mathcal{O}) \to K_3^c(\mathcal{O}_{\mathfrak{p}}) \to K_3(\mathbf{Z}/p^2) \to \mathbf{Z}/p,$$

(c) determine $\mathrm{Ch}_{\mathfrak{p}}(B) \neq 0$ in various cases by machine computation.

In the case $F = Q(\sqrt{-3})$, Tate has shown $K_2(\mathcal{O}) = 0$. Hence $K_3(\mathcal{O}) = H_3(\mathrm{St}_n(\mathcal{O})) = H_3(E_n(\mathcal{O}))$ for n large enough, where $E_n(\mathcal{O})$ is the group of $n \times n$ elementary matrices. The class $B \in H_3(E_n(\mathcal{O}))$ is represented as an explicit sum of 30 simplices in the bar resolution of $E_3(\mathcal{O})$, and the construction of B makes use of Riley's hyperbolic representation of the fundamental group of the complement of the figure eight knot [**R, M**]. As a cohomology class, the homomorphism $\mathrm{Ch}_{\mathfrak{p}} : K_3(\mathcal{O}) \to \mathbf{Z}/p$ comes from the diagram

$$\begin{array}{ccccc} K_3(\mathcal{O}) \to K_3^c(\mathcal{O}_{\mathfrak{p}}) & \to & K_3(\mathcal{O}/\mathfrak{p}^2) \cong K_3(\mathbf{Z}/p^2) & & \\ \downarrow & & \downarrow \qquad\qquad \downarrow & & \\ H_3(E(\mathcal{O})) & \to & H_3(E(\mathcal{O}/\mathfrak{p}^2)) \cong H_3(E(\mathbf{Z}/p^2)) & \xrightarrow[\mathrm{ch}]{} & \mathbf{Z}/p \end{array}$$

The explicit formula for the $E(\mathbf{Z}/p^2)$ invariant cocycle ch on a three simplex $g[a|b|c]$ in the bar resolution arises from examination of the standard cohomology class $\Delta \in H^2(\mathrm{GL}_n(\mathbf{Z}/p); M_n(\mathbf{Z}/p))$ of the extension

$$0 \to M_n(\mathbf{Z}/p) \to \mathrm{GL}_n(\mathbf{Z}/p^2) \to \mathrm{GL}_n(\mathbf{Z}/p) \to 1.$$

The class ch is a special case of a class constructed in $H^3(\mathrm{GL}_n(A/I^2); I^2/I^3)$ when A is semilocal with radical I such that A/I is finite.

Let $F = Q(\sqrt{-3})$ and recall [**Coh**] that a rational prime $p > 3$ is split iff -3 is a quadratic residue mod p. In this case solve $x^2 \equiv -3 \bmod p$. Then $\langle p \rangle = \mathfrak{p}\bar{\mathfrak{p}}$, where $\mathfrak{p} = \langle p, x + \sqrt{-3} \rangle$, $\bar{\mathfrak{p}} = \langle p, -x + \sqrt{-3} \rangle$, and $\mathcal{O}_{\mathfrak{p}} \cong \mathcal{O}_{\bar{\mathfrak{p}}} \cong \mathbf{Z}_p$.

THEOREM. *In the following cases $\Phi_{\mathfrak{p}} \otimes \mathbf{Z}_p$ is an isomorphism because* $\mathrm{Ch}_{\mathfrak{p}}(B) \neq 0$:

p	x	$\mathrm{Ch}_{\mathfrak{p}}(B)$ *for* $\mathfrak{p} = \langle p, x+\sqrt{-3}\rangle$
7	2	1
13	6	2
19	4	3
31	11	22
37	16	2
43	13	9
61	27	27
67	8	33
73	17	19

In general we have $\mathrm{Ch}_{\bar{\mathfrak{p}}}(B) = -\mathrm{Ch}_{\mathfrak{p}}(B)$ so there are nine more cases where $\mathrm{Ch}_{\mathfrak{p}} \neq 0$. The above examples were the only one computed. To make the computation for $\mathrm{Ch}_{\mathfrak{p}}$ we use the isomorphism $i_{\mathfrak{p}}\colon \mathcal{O}/\mathfrak{p}^2 \to \mathbf{Z}/p^2$ arising from

$$a + b\nu \to a + b((3+2x-x^2)/4x) \mod p^2,$$

where $\nu = (1+\sqrt{-3})/2$ and $a, b \in \mathbf{Z}$.

In [**Co**] Coleman uses the p-adic dilogarithm to define a homomorphism $D_p^*\colon K_3(C_p) \to C_p$, where C_p is a completion of the algebraic closure of Q_p. When $\mathcal{O}$ is the integers in the number field of mth roots of unity, he proves a regulator formula for $K_3(\mathcal{O})$ involving D_p^* and $L_p(2, \chi\omega^{-1})$, where χ has conductor m and ω is the Teichmüller character on $\mathbf{Z}_p^*$. In the case $m = 3$, Theorem 8.1 of [**Co**] suggests, after simplification, that we should have

$$L_p(2, \chi\omega^{-1}) \equiv -r_B i_{\mathfrak{p}}(x)\,\mathrm{Ch}_{\mathfrak{p}}(B) \mod p \tag{6}$$

for some rational number r_B depending only on B and having denominator prime to p. The factor r_B occurs because we only know $B \in \hat{K}_3(\mathcal{O})$ is some integer multiple of the generator. Machine computation verifies (6) holds for $r_B = 1/18$ in all cases of $\mathfrak{p} = \langle p, \pm x + \sqrt{-3}\rangle$ considered above.

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