CHARACTERIZATIONS OF MEASURES
WHOSE FOURIER-STIELTJES TRANSFORMS
VANISH AT INFINITY

BY RUSSELL LYONS[1]

We denote the class of complex Borel measures on the unit circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ by $M(\mathbf{T})$. Those measures μ whose Fourier-Stieltjes coefficients

$$\hat{\mu}(n) = \int_{\mathbf{T}} e^{-2\pi i n t} \, d\mu(t)$$

tend to 0 as $|n| \to \infty$ form the class R. The class of Borel sets E such that $\mu E = 0$ for all $\mu \in R$ is denoted U_0. We announce relations of R to certain subclasses of U_0. Full proofs will appear elsewhere.

THEOREM 1. *A measure μ lies in R if and only if $\mu E = 0$ for all $E \in U_0$.*

This result is a corollary of our principal result, Theorem 3 below. Let J be the class of measures concentrated on a U_0-set. An immediate consequence of Theorem 1 is

COROLLARY 2. *Each measure $\mu \in M(\mathbf{T})$ can be uniquely represented as $\mu = \mu_R + \mu_J$, where $\mu_R \in R$, $\mu_J \in J$, and $\mu_R \perp \mu_J$.*

Our method of proof of Theorem 1 involves a subclass of U_0, called W-sets. We first recall some facts from the theory of asymptotic distribution.

DEFINITION. A sequence $\{x_n\}_1^\infty \subset \mathbf{T}$ is said to have an *asymptotic distribution* if there exists $\nu \in M(\mathbf{T})$ such that for every arc $I \subset \mathbf{T}$ whose endpoints are not mass-points of ν,

$$\lim_{N \to \infty} \frac{1}{N} \#\{n \le N : x_n \in I\} = \nu I.$$

In this case, ν is called the asymptotic distribution of $\{x_n\}$. If ν is normalized Lebesgue measure, then $\{x_n\}$ is said to be *uniformly distributed.*

The principal theorem associated with asymptotic distribution usually goes by the name of "Weyl's criterion" (see [6, I, p. 142] for a proof).

WEYL'S CRITERION. *A sequence $\{x_n\}_1^\infty \subset \mathbf{T}$ has an asymptotic distribution if and only if for every $m \in \mathbf{Z}$,*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{-2\pi i m x_n} \quad \text{exists.}$$

In this case, the limit is $\hat{\nu}(m)$, where ν is the asymptotic distribution.

DEFINITION [5]. *If $\{x_n\}$ has an asymptotic distribution but is not uniformly distributed, then we say $\{x_n\}$ is Weyl-distributed. A Borel set $E \subset \mathbf{T}$ is called a W-set if there exists a strictly increasing sequence of positive integers $\{n_k\}$ such that for every $x \in E$, $\{n_k x\}$ is Weyl-distributed.*

Šreĭder [5] made the following unproved claim, which we have established.

THEOREM 3. *A measure μ lies in R if and only if $\mu E = 0$ for all W-sets E.*

The proof of Theorem 3 depends on Weyl's criterion and the well-known fact [6, II, p. 145] that if $\mu \in R$ and $|\nu|$ is absolutely continuous with respect to $|\mu|$, then $\nu \in R$. The crucial ingredient of our proof, however, is a new remarkable lemma.

LEMMA 4. *Given $\mu \in M(\mathbf{T})$ and $\{n_k\}_1^\infty \subset \mathbf{Z}$, there exists a subsequence $\{n_k'\} \subset \{n_k\}$ such that $\{n_k' x\}$ has an asymptotic distribution for $|\mu|$-almost all x.*

A sketch of the proof of Lemma 4 follows. Choose a subsequence $\{e^{-2\pi i n_k'' x}\}$ of $\{e^{-2\pi i n_k x}\}$ which weakly converges in $L^2(|\mu|)$. We may find a further subsequence $\{e^{-2\pi i n_k' x}\}$ such that if $f(x)$ is the weak limit of $e^{-2\pi i n_k'' x}$, then

$$\left\| \frac{1}{K} \sum_{k=1}^K e^{-2\pi i n_k' x} - f \right\|_{L^2(|\mu|)} = O(K^{-1/2}).$$

It follows that

$$\frac{1}{K^2} \sum_1^{K^2} e^{-2\pi i n_k' x} \to f(x) \quad \text{a.e } [\|\mu\|]$$

as $K \to \infty$, from which we deduce that also

$$\frac{1}{K} \sum_1^K e^{-2\pi i n_k' x} \to f(x) \quad \text{a.e. } [\|\mu\|].$$

By a diagonal argument, we may choose n_k' so that, likewise, for every $m \in \mathbf{Z}$,

$$\lim_{K \to \infty} \frac{1}{K} \sum_1^K e^{-2\pi i m n_k' x} \quad \text{exists a.e. } [\|\mu\|].$$

Weyl's criterion now implies that $\{n_k' x\}$ has an asymptotic distribution for $|\mu|$-a.e. x.

Theorem 3 extends in a natural way to all locally compact abelian groups.

Other attempts have been made in the past to characterize R by a theorem of the form of Theorem 3. Rajchman had conjectured (see [2, pp. 85–86]) that the class of H-sets, defined below, could be taken in place of the class of W-sets. On the other hand, Kahane and Salem [3, 4] asked whether U_0 contains the class of W^*-sets, to be defined presently. As we shall see, this amounts to the question of whether W^*-sets can replace W-sets in Theorem 3. However, neither H-sets nor W^*-sets can replace W-sets (see Theorems 7, 6 below).

DEFINITION. *A Borel set $E \subset \mathbf{T}$ is a W^*-set (or nonnormal set) if there exists $n_k \uparrow \infty$ such that for every $x \in E$, $\{n_k x\}$ is not uniformly distributed.*

Since W-sets are clearly W^*-sets, Theorem 3 implies

THEOREM 5. *If $\mu E = 0$ for all $E \in W^*$, then $\mu \in R$.*

Actually, Theorem 5 has an elementary proof. Briefly, if $\mu \notin R$, take $\hat{\mu}(n_k) \to \alpha \neq 0$. Let

$$E = \{x \colon \{n_k x\} \text{ is not uniformly distributed}\}.$$

Then one may show that

$$\lim_{K \to \infty} \int_E \frac{1}{K} \sum_{k=1}^{K} e^{-2\pi i n_k x} \, d\mu(x) = \alpha \neq 0,$$

whence $|\mu|E \neq 0$. Since E is a W^*-set, Theorem 5 follows. We remark that the ideas here are those necessary to prove Theorem 3 using Lemma 4.

Thus, we see that Kahane's and Salem's question of whether $W^* \subset U_0$ is equivalent to asking if W^*-sets characterize R in the manner that W-sets do. As we have said, this is untrue. We may, in fact, give precise conditions on the rate of decay of $\hat{\mu}$ at infinity in order that μ annihilate all W^*-sets corresponding to a sequence $\{n_k\}$ with $\liminf_{k \to \infty} n_{k+1}/n_k > 1$, the so-called *lacunary W^*-sets*.

THEOREM 6. *If $\phi(n)$ is a decreasing function on the nonnegative integers such that*

(1)
$$\sum_{n=2}^{\infty} \frac{\phi(n)}{n \log n} < \infty,$$

and if μ is a positive measure such that

(2)
$$|\hat{\mu}(n)| \leq \phi(|n|)$$

for all n, then $\mu E = 0$ for all lacunary W^*-sets. However, if ϕ is decreasing and (1) fails, then there exists a positive measure μ satisfying (2) and such that μ is concentrated on the set of nonnormal numbers base 2, i.e., on the lacunary W^*-set

$$E = \{x \in \mathbf{T} \colon \{2^{k-1} x\}_{k=1}^{\infty} \text{ is not uniformly distributed}\}.$$

The first part of Theorem 6 is a slight extension of a result of Baker [1]. The proof of the second part involves infinite convolution measures.

We now turn to Rajchman's conjecture about H-sets.

DEFINITION. A Borel set $E \subset \mathbf{T}$ is called an *H-set* if there exists $n_k \uparrow \infty$ and a nonempty open arc $I \subset \mathbf{T}$ such that for all $x \in E$ and all k, $n_k x \notin I$.

It has long been known that $H \subset U_0$. Rajchman's conjecture asserts that also $\mu \in R$ if $\mu E = 0$ for all H-sets E. A counterexample to this assertion is given by the next result.

THEOREM 7. *Let μ be the Riesz product*

$$d\mu = \prod_{k=1}^{\infty} \left(1 + a_k \cos 2\pi (n_k x + \phi_k)\right) dx,$$

where $-1 \leq a_k \leq 1$. If $n_{k+1}/n_k \to \infty$, then $\mu E = 0$ for all H-sets E.

Note that if $a_k \not\to 0$, then $\mu \notin R$. It can also be shown that Riesz products not belonging to R are concentrated on W-sets and hence belong to J. Thus, Riesz products μ are pure: $\mu \in R$ or $\mu \in J$. The same holds for infinite convolutions of discrete measures, extending the Jessen-Wintner purity law.

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