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Lagrangian analysis and quantum mechanics, a mathematical structure related to asymptotic expansions and the Maslov index, by Jean Leray, the MIT Press, Cambridge, Mass., 1982, xvii + 271 pp., \$35.00. ISBN 0-2621-2087-9

Semi-classical approximation in quantum mechanics, by V. P. Maslov and M. V. Fedoriuk, Mathematical Physics and Applied Mathematics, vol. 7, D. Reidel Publishing Company, Dordrecht:Holland/Boston:U.S.A./London:England, 1981, ix + 294 pp., Cloth Dfl. 125.00/U.S. \$66.00. ISBN 9-0277-1219-0

1. It is a fundamental principle in quantum mechanics that, when the time and distance scales in a system are large enough relative to Planck's constant h , the system will approximately obey the laws of classical, Newtonian mechanics. To confound the uninitiated who think that physical constants are immutable this is usually rephrased: in the limit $h \rightarrow 0$ quantum mechanics tends to Newtonian mechanics. In either form this principle says very little. Quantum mechanics would not be widely accepted if it did not predict that boulders and freight trains obey Newton's laws. Quasi-classical approximations express this limiting behavior in more useful ways, through formulas for expectations, energy levels, etc. which are asymptotic to the exact formulas as $h \rightarrow 0$. The paradigm of such a formula is Bohr's energy quantization law. Bohr actually deduced this *before* the "exact" formula was introduced by Schrödinger. Nonetheless, Bohr's law can be rederived and generalized as a quasi-classical approximation. Quasi-classical approximations for problems with more than one degree of freedom are rather new. The first book to deal with them in some generality was Maslov's remarkable monograph [8]. In his preface to the French translation of [8] Jean Leray noted that a mathematician reading it would read much more between the lines than on them. *Quasi-classical approximations in quantum mechanics* (= QAQM) and *Lagrangian analysis and quantum mechanics* (= LAQM) are not so much sequels to [8] as systematic efforts to fill in those missing lines. (In this review we use the exact English translation of the Russian title of the book of Maslov and Fedoriuk. "Quasi-classical" and "semi-classical" appear to be used equally often in the English literature.)

To discuss the contents of QAQM and LAQM and relate them to other works on this subject we will need to explain a bit how quasi-classical approximation works. One begins with a linear partial differential equation which can be put in the form

$$(1) \quad 0 = Pu = \sum_{j=0}^N \epsilon^j P_j(x, \epsilon D)u,$$

where $P_j(x, \xi)$ is a polynomial in the second variable, $D = (\partial/i\partial x_1, \dots, \partial/i\partial x_n)$ as usual, and ϵ is a parameter. Important simple examples are

(2a) Schrödinger's equation

$$\frac{\epsilon}{i} \frac{\partial u}{\partial t} = \frac{-\epsilon^2}{2m} \Delta u + Vu.$$

(2b) Schrödinger's equation for energy levels

$$Eu = \frac{-\epsilon^2}{2m} \Delta u + Vu.$$

(2c) The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2(x) \Delta u.$$

In (2a) and (2b) ϵ is the constant $\hbar = (2\pi)^{-1}h$ of the physics literature. To put (2c) in the form above you must multiply by h^2 , but that is permissible. Then one looks for approximate solutions to (1) in a special form,

$$(3) \quad u = e^{iS(x)/\epsilon} (a_0(x) + \epsilon a_1(x) + \dots + \epsilon^M a_M(x))$$

where the functions $S(x), a_1(x), \dots, a_M(x)$ do not depend on ϵ , and $S(x)$ is real-valued. Where (3) comes from is a long story whose end we could not hope to reach here. The point is that (3) works. Applying the operator P in (1) to this u and equating the coefficients of $\epsilon^j, j = 0, 1, \dots, M + 1$, to zero, one gets a sequence of equations beginning with

$$(4) \quad P_0 \left(x, \frac{\partial S}{\partial x}(x) \right) = 0 \quad (\text{from } \epsilon^0)$$

and followed by $M + 1$ linear, first order partial differential equations which can be solved successively to determine a_0, \dots, a_M . The equation (4) is solved by the method of characteristics: one prescribes S on a hypersurface H and chooses the normal derivative of S on H so that (4) holds on H . Then one solves

$$(5) \quad \dot{x} = \frac{\partial P_0(x, \xi)}{\partial \xi}, \quad \dot{\xi} = -\frac{\partial P_0(x, \xi)}{\partial x}$$

with $x(0, y) = y$ and $\xi(0, y) = \partial S(y)/\partial x$ for $y \in H$. The desired function S satisfies

$$(6) \quad \frac{\partial S}{\partial x}(x(s, y)) = \xi(s, y)$$

and one recovers S from (6). One can explain (6) by the invariance of the canonical 2-form on $T^*(\mathbf{R}^n)$ under Hamiltonian flows or as a magical computational fact, according to personal preference.

At this point one has

$$Pu = e^{iS/\epsilon}g(x, \epsilon)$$

where g and its derivatives are $O(\epsilon^{M+2})$ and it is rather simple to show that u is asymptotic to the true solution to the corresponding problem as $\epsilon \rightarrow 0$. For the Schrödinger equation (2a) and the wave equation (2c) the natural problem is the Cauchy problem with Cauchy data at $t = 0$ equal to those of u . In this case one concludes u is asymptotic because the Cauchy problem is well posed.

The procedure described thus far goes under several names: the WKB method, the geometric optics expansion, and others, but we won't attempt to trace its lineage here. Suffice it to say that it was well understood prior to 1960. The issue in all subsequent work on these problems has been what to do about the following bug in the method: everything breaks down if the mapping

$$\Phi : (s, y) \mapsto x(s, y)$$

fails to be invertible. This has the effect of limiting constructions of asymptotic solutions for Cauchy problems to short time intervals, and it is completely fatal when one attacks eigenvalue problems like (2b). The starting point for overcoming the difficulty is the observation that $(x(s, y), \xi(s, y)), y \in H, -\infty < s < \infty$, is a smooth n -dimensional manifold Λ in \mathbf{R}^{2n} . The canonical form $\sum_{i=1}^n d\xi^i \wedge dx^i$ vanishes on the tangent space of Λ , and it has become standard to call such manifolds “lagrangian”. In the case of Schrödinger’s equation the equations (5) are just

$$(7) \quad \begin{aligned} \dot{x}_i &= \frac{1}{m} \xi_i, & i = 1, \dots, 3, & \quad \dot{x}_0 = 1, \\ \dot{\xi}_i &= -\frac{\partial V}{\partial x_i}, & i = 1, \dots, 3, & \quad \dot{\xi}_0 = 0, \end{aligned}$$

and, if we identify x_0 with time and ξ_0 with energy, we see that Λ is just a set of classical Newtonian trajectories for our system in phase space (with time and energy included). The noninvertibility of Φ corresponds to intersections in the trajectories when we project Λ down to configuration space. In optics the image of the set where the jacobian of Φ vanishes was known as the “caustic” set, since geometric optics predicts the intensity of light will be infinite there, and this term has also become standard.

To build asymptotic solutions near the caustic set one needs to enlarge the class of functions considered in (3). This is done by making the new ansatz:

$$(8) \quad u = \int_{\mathbf{R}^k} e^{iS(x,\alpha)/\epsilon} (a_0(x, \alpha) + \dots + \epsilon^M a_M(x, \alpha)) d\alpha.$$

One wants these functions to match up with those of (3) when the projection of Λ onto x -space is nonsingular. Since all contributions to u in (8) from integrals

over sets where $\partial S(x, \alpha)/\partial \alpha \neq 0$ are $O(\epsilon^m)$ for all m , one decides that

$$\Lambda_0 = \left\{ \left(x, \frac{\partial S}{\partial x}(x, \alpha) \right) : \frac{\partial S}{\partial \alpha}(x, \alpha) = 0 \right\}$$

should be a subset of Λ , and asks that $\partial S(x, \alpha)/\partial \alpha = 0$ define a smooth n -dimensional manifold in $\mathbf{R}^n \times \mathbf{R}^k$. This idea or at least the systematic exploitation of it is due to Maslov. There are several ways to introduce $S(x, \alpha)$ so that Λ_0 is a subset of Λ . The method of [8] and QAQM is based on the fact that locally one can always choose n of the variables (x, ξ) so that L has a nonsingular projection onto these variables; that of Duistermaat and Hörmander in [3, 5 and 2] is based on the fact that, after a change of variables induced by a local change of variables in x -space, Λ has a nonsingular projection onto the ξ -variables. Either way one finds an open cover $O_i, i \in I$, of Λ and an S_i with the desired properties for each O_i in the cover. If O_i has a nonsingular projection onto x -space, $(\partial^2 S/\partial \alpha_i \partial \alpha_j)$ must be nonsingular and one can solve $\partial S/\partial \alpha = 0$ for $\alpha(x)$ by the implicit function theorem. In this case the integral in (8) has a very well-known asymptotic expansion by the method of stationary phase

(9)

$$u \sim \epsilon^{k/2} \frac{e^{i(\pi\sigma/4 + S(x, \alpha(x)))/\epsilon}}{|\det \partial^2 S(x, \alpha(x))/\partial \alpha^2|^{1/2}} (a_0(x, \alpha(x)) + \epsilon f_1(x) + \epsilon^2 f_2(x) + \dots),$$

where σ is the signature of $\partial^2 S/\partial \alpha^2$. Thus (8) *does* reduce to (2) with the correct S for our problem when Λ projects onto x -space nonsingularly.

The problem now is how to choose $a_0(x, \alpha), \dots, a_M(x, \alpha)$ so that (8) is an asymptotic solution of (1). The most naive way to approach this would be to work a little harder in the construction of $S(x, \alpha)$ so that

$$P_0 \left(x, \frac{\partial S}{\partial x}(x, \alpha) \right) = 0$$

holds identically in (x, α) and not just on Λ . Then one could solve for the functions $a_0(x, \alpha), \dots, a_M(x, \alpha)$ exactly as in the geometric optics expansion. This actually works as long as \dot{x} in (5) never vanishes on Λ . Hence it suffices for the Schrödinger and wave equations. However, it is not adequate for the Schrödinger eigenvalue problem (2b) where it always fails when $n = 1$, and this approach has never been taken in the literature to date.

The approach in QAQM is that of [8], where some of the x -variables in $a_i(x, \alpha)$ are suppressed and the parameters α are introduced in such a way that

$$a(x, \alpha, h) = \sum_{i=0}^M \epsilon^i a_i(x, \alpha)$$

can be identified with a function *on* Λ . Then (8) is interpreted as an operator K , the “precanonical operator”, acting on $a(x, \alpha, \epsilon)$. To make u in (8) satisfy the differential equation to “order ϵ^2 ” one finds the operator Q_0 such that $PK = K(Q_0 + O(\epsilon^2))$ and requires $Q_0 a_0 = 0$. This amounts to the same thing

as simply applying P to u and “differentiating under the integral sign”. This gives

$$(10) \quad Pu = \int e^{iS/\epsilon} \left(P_0 \left(x, \frac{\partial S}{\partial x} \right) + R_0 + O(\epsilon^2) \right) a(x, \alpha, \epsilon) d\alpha$$

where R_0 is a first order differential operator in x with coefficients of order ϵ . Since $P_0(x, \partial S/\partial x) = 0$ when $\partial S/\partial \alpha = 0$, one can integrate by parts in α and replace $P_0(x, \partial S/\partial x)$, by a first order differential operator R'_0 in α with coefficients of order ϵ . The operator $R_0 + R'_0$ is Q_0 . When one identifies Λ with its pre-image in (x, α) -space, Q_0 lifts to a well-defined differential operator on Λ (its top order part is just the Hamiltonian vector field from (5)) and it makes sense to require $Q_0 a_0 = 0$ on Λ .

At this point enough machinery has been introduced to make it possible to study the local behavior of quasi-classical approximations near the caustic sets. However, in some instances, particularly the eigenvalue problems (2b), one wants global constructions, where the approximations u_i associated with different O_i 's in the cover of Λ are patched together to give a well-defined function on \mathbf{R}^n . To do this Maslov and Fedoriuk follow [8] and introduce the “canonical operator” by adding a factor for each O_i before $a(x, \alpha, \epsilon)$ under the integral sign in (8). In the case that $O_i \cap O_j$ has a nonsingular projection onto x -space, these factors cancel out the change in

$$\left| \det \frac{\partial^2 S}{\partial \alpha^2} \right|^{-1/2} \exp \left(\frac{i\pi}{4} \operatorname{sgn} \frac{\partial^2 S}{\partial \alpha^2} \right)$$

as one goes from O_i to O_j (see (9)). Modulo some cohomological restrictions which will be discussed shortly, this makes it possible to define the canonical operator on all of $C_0^\infty(\Lambda)$, and the new Q_0 becomes a globally defined operator on Λ . An alternate (but equivalent) approach used in the expositions of the theory that came after [8] is to try to define the lowest order part in ϵ , or “principal symbol” of $u(x)$. As one might (conceivably) guess from (9) this principal symbol has to be defined as $\frac{1}{2}$ -density on Λ with values in a \mathbf{Z} -bundle over Λ (cf. [2 and 12]) or as a “ $\frac{1}{2}$ -form on Λ ” (cf. [4]). Readers who want to be involved in as little geometry or topology as possible may prefer QAQM; those who like geometry will prefer $\frac{1}{2}$ -densities and those who like both geometry and algebra will like $\frac{1}{2}$ -forms. All three approaches lead to the same point: one sees how to define global quasi-classical approximation pieced together from the functions in (8).

This lengthy introduction will end with a description of what one can do by this method in the problem (2b). For the Schrödinger eigenvalue problem one looks for cases where Λ is compact, and one immediately encounters the cohomological obstruction mentioned earlier. For any closed curve C on Λ , the phase $S/2\pi\epsilon$ must increase by the “action” $(2\pi\epsilon)^{-1} \int_C \xi \cdot dx$ as one goes around C , since $\nabla S = \xi$ on Λ . Note that, since $\sum d\xi_i \wedge dx_i$ vanishes on Λ , the action depends only on the cohomology class of C . At the same time one has to consider the cumulative effect of the exponential factors analogous to $\exp(i\pi/4) \operatorname{sgn}(\partial^2 S/\partial \alpha^2)$ in (9). One of the main points of [8], which was partially anticipated by Keller [6] and further explained by Arnol'd [1], was

that the total contribution of these “phase shifts” as one goes around C is given by a \mathbf{Z}_4 -valued cohomology class of Λ —the Maslov index $\gamma(C)$. When the action and $\gamma/4$ cancel modulo \mathbf{Z} , the canonical operator becomes well defined on $C_0^\infty(\Lambda)$ modulo terms of order ϵ^2 . Thus after a simple analysis of the equation $Q_0 a_0 = 0$ on Λ one arrives at Maslov’s generalization of Bohr’s Law: given a homology basis C_1, \dots, C_l for Λ , if

$$(11) \quad \frac{1}{2\pi\epsilon} \int_{C_i} \xi \cdot dx - \frac{\gamma(C_i)}{4} \in \mathbf{Z}$$

for $i = 1, \dots, l$, then one can construct a function u such that

$$Eu + \frac{\epsilon^2}{2m} \Delta u - Vu = O(\epsilon^2).$$

Since the norm of the resolvent of a selfadjoint operator at $\lambda = E$ is the distance from E to the spectrum, one concludes that there is an exact energy level \hat{E} satisfying $|E - \hat{E}| = O(\epsilon^2)$. On the other hand, since each of the equations in (11) is going to hold only for a discrete set of ϵ , one also concludes that none of this is going to be very meaningful unless we have an l -parameter family of Λ ’s all invariant under (7) so that, varying Λ we can make (11) hold identically in ϵ . The only examples where (7) has such large families of compact lagrangian invariant manifolds are those where it is completely integrable. This severely restricts the applicability of this quasi-classical approximation. Moreover, completely integrable systems in classical mechanics usually go over into separable systems in quantum mechanics, and there are simpler methods of quasi-classical approximation for those. Still, interesting nonseparable but completely integrable systems do occur. The periodic Toda lattice is one example.

2. QAQM. The monograph of Maslov and Fedoriuk gives a detailed presentation of the theory sketched in §1. Chapters 1–9 are devoted to the general theory of the canonical operator and the Maslov index. The construction of asymptotic solutions to oscillatory initial value problems for equations like the Schrödinger and the acoustic wave equations is given in Chapters 10–12. Chapter 13 contains the generalized Bohr Law. The operators $P(x, \epsilon D, \epsilon)$ considered in this presentation are more general than those in §1 in two respects. Firstly they are not differential, but pseudo-differential with the analogous dependence on the parameter ϵ and may have quite general symbols. The operator sending $f(x)$ to $f(x + \epsilon t)$ is permitted (though not the one sending $f(x)$ to $f(x + t)$ — as is claimed in an oversight in the introduction). Secondly matrices of such operators are considered. The passage to matrices only involves changes at the level of geometric optics, and it substantially enhances the applicability of the methods. An application to the relativistic Dirac equation is given in the final Chapter 14. On the other hand, in all the applications given, $P(x, \epsilon D, \epsilon)$ is differential. Little would have been lost if the pseudo-differential P ’s had been suppressed in this exposition.

The style of the presentation is consistently computational and local coordinates abound. One of the strengths of the Russian mind seems to be the ability to withstand notation of stupefying complexity. Readers unaccustomed to this

style will find that the notation gets in their way: conceptually simple computations via stationary phase like Lemmas 6.2 and 6.3 turn into a soup of indices, and a formula like (10) won't fit on less than four lines (see (5.9) of QAQM). Nonetheless from a pedagogical point of view there is much to be said for the explicit form of Maslov and Fedoriuk's constructions. The presentation of the general theory in Chapters 1–9 is also liberally leavened with illustrative examples. In the applications, Theorem 12.5 gives a particularly good illustration of how the index contributes to the asymptotic form of solutions to oscillatory initial value problems. *More* examples would have been welcome in Chapter 13: the only explicit asymptotic eigenvalue computation is done for the Laplacian on S^n .

In sum *Quasi-classical approximations in quantum mechanics* is a good exposition of Maslov's theory, which amply clarifies the original presentation in [8]. It assumes very little knowledge of symplectic differential geometry, and of all the treatments available it gives the most explicit computations.

The translation of QAQM deserves a grade of approximately B. The English has a few lapses ("Now suppose A be a real...matrix" (p. 30)), and definite articles don't always turn up where you expect them, but this merely gives the book an agreeably foreign flavor. Less pardonable is the translators' use of the unlovely invention "inerdex" for the index of a quadratic form. The book does have quite a few misprints. Fortunately some of the more confusing ones occur in the statements of results from [1] in Chapter 7, and the reader can correct them by going directly to [1].

As is inevitable in a translation of a six-year-old book, QAQM is a little out of date. Some of the most interesting quasi-classical computations have been made in boundary value problems for (bicharacteristically) convex and concave obstacles. Here one gets formulas in scattering theory which are not plagued by the restrictions that attend the asymptotic eigenvalue computations. This work was just beginning when QAQM appeared in 1976. For the student who wishes to pursue this we would recommend M. E. Taylor's [11], followed by [9, 13 and 10].

3. LAQM. In the Preface to LAQM, Leray tells us that in 1967 he was asked by Arnol'd for his thoughts on Maslov's work, and that the present work is his answer to this question. He points out, in the Preface, an apparent contradiction in Maslov between the facts that, on the one hand, the differential equations (1) of §1 have solutions (3), which are to be thought of as holding asymptotically as $\epsilon \rightarrow 0$ and that, on the other hand, when the smooth manifold Λ introduced in §1 is compact, it is subject to certain quantum conditions (11) which require ϵ to be fixed. He then claims that the way out of this apparent dilemma is the introduction of a

"new mathematical structure, Lagrangian Analysis, which requires the datum of a constant ν_0 ($= 2\pi i/h = i/\hbar$) and is based on symplectic geometry...

*It introduces ν_0 for defining lagrangian functions...in the same manner as Planck introduced \hbar for describing the spectrum of the blackbody. Thus the book could be entitled *The introduction of Planck's constant into mathematics.*"*

We will have more to say on this claim at the end of this review.

The structure of lagrangian analysis is a very complicated one. Some idea of it can be gotten by reading through Leray's Poincaré Symposium paper [7]. Fundamental to his approach are the notions of *lagrangian function* and *lagrangian operator*. These may be explained as follows:

We let $\mathfrak{B}(\mathbf{R}^n)$ be the algebra of \mathbf{C} -valued functions on $\mathbf{R}^n \times (0, 1]$ which are C^∞ on $\mathbf{R}^n \times \{h\}$ for all $h \in (0, 1]$ and which satisfy

$$\sup_{(x,h)} \left| p \left(x, ih \frac{\partial}{\partial x} \right) f(x, h) \right| < \infty$$

for all polynomials p , and we let $\mathcal{C}(\mathbf{R}^n)$ be $\mathfrak{B}(\mathbf{R}^n)$ modulo the ideal of all $f \in \mathfrak{B}(\mathbf{R}^n)$ such that $h^{-N}f(x, h) \in \mathfrak{B}(\mathbf{R}^n)$ for all N . $\mathcal{C}(\mathbf{R}^n)$ is the algebra of all *asymptotic equivalence classes* on \mathbf{R}^n . Differential operators of the form $p(x, ih\partial/\partial x)$, p a polynomial, operate on $\mathcal{C}(\mathbf{R}^n)$, and the map $f \mapsto \int_{\mathbf{R}^n} f(x, h) dx$ sends $\mathcal{C}(\mathbf{R}^n)$ to $\mathcal{C} = \mathcal{C}(\mathbf{R}^0)$ and is called the *asymptotic integral*. Among the members of $\mathcal{C}(\mathbf{R}^n)$ are the so-called *formal functions with compact support* $\mathfrak{F}_0(\mathbf{R}^n)$, functions of the form

$$\sum_{j=1}^m \sum_{r=0}^{\infty} h^r \alpha_{j,r} e^{2\pi i \varphi_j/h},$$

where the α 's are in $C_0^\infty(\mathbf{R}^n)$ and the φ 's are in $C^\infty(\mathbf{R}^n)$.

Let $\mathbf{R}^{2n} = \mathbf{R}^n \oplus \mathbf{R}^n$ have its usual structure of symplectic vector space with symplectic inner product $[\cdot, \cdot]$, let Λ be a lagrangian submanifold of \mathbf{R}^{2n} , and let $\tilde{\Lambda}$ be the simply connected covering space of Λ . Let $\text{Sp}_2(n)$ be the 2-fold covering group of the group of symplectic $2n \times 2n$ matrices, sometimes called the *metaplectic group* $\text{Mp}(n, \mathbf{R})$, and let $\mathfrak{R}(n)$ denote the double covering of the space of symplectic bases of \mathbf{R}^{2n} . Then $\text{Sp}_2(n)$ acts on $\mathfrak{R}(n)$. Each $R \in \mathfrak{R}(n)$ defines a linear isomorphism \bar{R} of \mathbf{R}^{2n} with itself; we let Σ_R be the set of $(x_0, \xi_0) \in \Lambda$ such that the projection π_R of Λ to \mathbf{R}^n given by the composition of \bar{R} with $(x, \xi) \mapsto x$ is singular at (x_0, ξ_0) , and we let $\tilde{\Sigma}_R$ be the inverse image of Σ_R under the projection of $\tilde{\Lambda}$ onto Λ . Formal functions can then be defined with respect to a given $R \in \mathfrak{R}(n)$ on open subsets of $\tilde{\Lambda}$. Moreover, $\mathfrak{F}_0(\tilde{\Lambda} \setminus \tilde{\Sigma}_R, R)$, the formal functions defined with respect to $R \in \mathfrak{R}(n)$ with compact support in the complement $\tilde{\Lambda} \setminus \tilde{\Sigma}_R$ of $\tilde{\Sigma}_R$ in $\tilde{\Lambda}$, project into $\mathfrak{F}_0(\mathbf{R}^n)$ via a map Π_R , which is defined simply by summation over $\tilde{\pi}_R^{-1}(x)$, $x \in \mathbf{R}^n$, where $\tilde{\pi}_R = \pi_R \circ p$. These objects are set up in such a way that everything is as covariant with respect to $\text{Sp}_2(n)$ as it possibly can be. In particular, every $S \in \text{Sp}_2(n)$ acts in a natural way on $\mathcal{C}(\mathbf{R}^n)$ via the Maslov index and an asymptotic integral formula which holds for almost all S . Moreover S can be made to send $\mathfrak{F}_0(\tilde{\Lambda} \setminus \tilde{\Sigma}_R \cup \tilde{\Sigma}_R, R)$ to $\mathfrak{F}_0(\tilde{\Lambda} \setminus \tilde{\Sigma}_{SR} \cup \tilde{\Sigma}_{SR}, SR)$ in such a way that this action intertwines Π_R and Π_{SR} , and can be extended to noncompactly supported functions.

In a similar way, the set $\mathfrak{D}(n)$ of asymptotic polynomial differential operators on \mathbf{R}^n are acted on by $\text{Sp}_2(n)$ in such a way that $(S \cdot D)f = SDS^{-1}f$ for such an operator D and for $S \in \text{Sp}_2(n)$, $f \in \mathcal{C}(\mathbf{R}^n)$. Indeed, the $D \in \mathfrak{D}(n)$ correspond bijectively to polynomials a_D on \mathbf{R}^{2n} in such a way that $a_{S \cdot D} = a_D \circ \tilde{S}^{-1}$, where \tilde{S} is the element of $\text{Sp}(n)$ lying beneath S . The polynomial a_D

is a sort of symmetrized symbol of D . Then $D \in \mathcal{D}(n)$ can be made to act on $\mathcal{F}(\tilde{\Lambda} \setminus \tilde{\Sigma}_R, R)$ in a way covariant with the projection Π_R and with the actions of $\text{Sp}_2(n)$ on all of the above. Finally, all of this can be extended to operators with quite arbitrary symmetrized symbols; these operators are called *lagrangian operators*.

At last we come to the introduction of Planck's constant. The trouble with the foregoing is that one is interested not in lagrangian functions on $\tilde{\Lambda}$, but in lagrangian functions on Λ itself. Let $\gamma \in \pi_1(\Lambda)$, realized as the group of deck transformation of $\tilde{\Lambda}$ over Λ , and let $f \in \mathcal{F}(\tilde{\Lambda} \setminus \tilde{\Sigma}_R, R)$. Unfortunately $f \circ \gamma^{-1} \notin \mathcal{F}(\tilde{\Lambda} \setminus \tilde{\Sigma}_R, R)$. This defect can be remedied as follows: choose $h_0 \in (0, \infty)$ and define γf to be the asymptotic object $\exp(2\pi i(h^{-1} - h_0^{-1})c_\gamma)(f \circ \gamma^{-1})$, where $c_\gamma = \frac{1}{2} \int_{\tilde{\gamma}} [z, dz]$, $\tilde{\gamma}$ being a loop in $\tilde{\Lambda}$ representing γ . In this way we get $\pi_1(\Lambda)$ to operate on $\mathcal{F}_0(\tilde{\Lambda} \setminus \tilde{\Sigma}_R, R)$, and this operation commutes with all actions of $\text{Sp}_2(n)$. Lagrangian functions on Λ are then just $\pi_1(\Lambda)$ -invariant lagrangian functions on $\tilde{\Lambda}$, and lagrangian operators operate on these. *This constant h_0 may be identified with Planck's constant* in many applications of the theory.

As for these applications, Leray gives us, after some general considerations, just the same examples considered by other authors, *viz.* the Schrödinger, Klein-Gordon, and Dirac equations. It is a great deal to ask the reader to digest the immense machinery developed in this book just to be led back to the same problems one has always dealt with, particularly since the style of the book is severe and didactic in the extreme (the severity is not helped by the stiff translation). But the effort may be worthwhile. Lagrangian analysis is a beautiful theory and may lead to a deeper understanding of the subject. Indeed, one wonders if the theory, which is strictly one covariant under the linear metaplectic group, is extendible to a theory on metaplectic manifolds which is covariant under all metaplectic diffeomorphisms. Perhaps not. It is known that there are formidable roadblocks along the way to this end. Leray's book is certainly a major milestone on the journey.

However, there remains the question raised at the beginning of this section: to what extent does Leray's formalism resolve the problem of reconciling the method of asymptotic expansions, which depends on a varying parameter $h \rightarrow 0$, and the fact that the Maslov quantization conditions specify that one work only on certain lagrangian submanifolds of the characteristic variety of the symbol of an operator, and these quantization conditions involve a fixed h ? It seems to us that Lagrangian analysis, as developed in LAQM, only goes as far as the work of Duistermaat [2], Guillemin and Sternberg [4], and Weinstein [12]. That is, in looking for solutions to equations (1) of §1 which hold modulo ϵ^2 , Leray is just solving the characteristic equation to obtain the lagrangian submanifold Λ and then solving the first transport equations on Λ , all this in a completely $\text{Sp}_2(n)$ covariant way. As we described earlier, what Maslov and Fedoriuk do in QAQM, in the completely integrable case, is to allow Λ to vary with h . In eigenvalue problems, this allows them to obtain an asymptotic series for eigenvalues, which Leray does not seem to discuss. For this and other reasons, QAQM seems closer to physics than LAQM.

Finally, we mention that QAQM has a small, but serviceable index, while LAQM lacks one altogether.

