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The theory of Eisenstein systems, by M. Scott Osborne and Garth Warner, Academic Press, New York, 1981, xiii + 385 pp., $55.00. ISBN 0-1252-9250-3

The Laplace-Beltrami operator on the upper half-plane with respect to the hyperbolic metric is

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The arithmetic interest of the eigenfunctions of Δ invariant under the modular group $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ and its congruence subgroups was signalled by Maass [**17**], who was inspired by earlier work of Hecke. If $\gamma \in \mathrm{GL}(2, \mathbf{Q})$ then $\Gamma_\gamma = \gamma^{-1}\Gamma\gamma \cap \Gamma$ is of finite index in Γ. Thus if $\det \gamma > 0$ so that γ also acts as a fractional linear transformation on the upper half-plane one can introduce the operator

$$T_\gamma : f \to \sum_{\delta \in \Gamma_\gamma \backslash \Gamma} f(\gamma\delta z), \qquad \operatorname{Im} z > 0.$$

It is called a Hecke operator. It commutes with Δ, and acts on its eigenspaces. The study of these operators and of those appearing in Hecke's work promises to be of considerable importance for diophantine problems, in particular for the investigation of the Dirichlet series to which the names of Artin and Hasse-Weil are attached. However the spectral theory of Δ on Γ-invariant functions is a purely analytic problem, of interest in its own right for any discrete subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$ whose fundamental domain has finite volume. If the quotient of the upper half-plane by Γ is compact the spectrum is discrete, but otherwise there is a continuous spectrum and the corresponding eigenfunctions are called Eisenstein series.

If the quotient is not compact there are cusps. By way of illustration we may assume that ∞ is a cusp. This means that Γ contains a subgroup of the form

$$\Gamma_0 = \left\{ \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbf{Z} \right\}$$

and that a part of the fundamental domain can be taken to be

$$\{z = x + iy \mid -a/2 < x \leqslant a/2, y > b\}.$$

Here a and b are positive real numbers, and for convenience we take $a = 1$.

Then a function ψ invariant under Γ has a Fourier expansion

$$\psi(x, y) = \sum_{n=-\infty}^{\infty} \psi_n(y) e^{2\pi i n x},$$

and $\psi_0(y)$ is called the constant term at ∞. If the constant term at all cusps is 0 then ψ is called a cusp form. If ψ is an eigenfunction of Δ then

$$\psi_n'' - 4\pi^2 n^2 \psi_n = \frac{\lambda}{y^2}\psi_n,$$

so that

$$(1) \qquad \psi_0 = \alpha y^{1/2+s} + \beta y^{1/2-s},$$

with $s^2 - \frac{1}{4} = \lambda$. For $n \neq 0$ the equation has an exponentially increasing solution, which can play no role in the spectral theory, and an exponentially decreasing solution, which is thus square-integrable in a neighborhood of the cusp for the invariant volume is $dxdy/y^2$. As a consequence one can expect that the spectrum of Δ in the space of cusp forms will be discrete. This was proved by Roelcke [**18**].

On the orthogonal complement, with respect to the inner product defined by the invariant area $dxdy/y^2$, of the space of cusp forms functions are controlled by their constant term. Thus on this space Δ can be regarded as a perturbation of the operator $y^2 d^2/dy^2$ on the half-line $y \geqslant 1$ with respect to the measure dy/y^2, or rather of the direct sum of r such operators if there are r cusps. Consequently there should be an r-fold continuous spectrum of Lebesgue type on $-\infty < \lambda \leqslant -\frac{1}{4}$ together with a finite set of discrete eigenvalues.

The present problem has a special feature: the perturbed eigenfunctions can be constructed explicitly. Observe that $F(z, s) = y^{1/2+s}$, $z = x + iy$, is an eigenfunction of Δ as are all its translates by elements of Γ. The series

$$E(z, s) = \sum_{\Gamma_0 \backslash \Gamma} F(\gamma z, s)$$

converges for $\operatorname{Re} s > \frac{1}{2}$ and gives an eigenfunction of Δ. One can build the analogous function for each cusp, try to analytically continue it to $\operatorname{Re} s = 0$, and in this way obtain the eigenfunctions for the continuous spectrum. The problem was posed by Roelcke, and solved by him for congruence subgroups, for which these Eisenstein series reduce to classical series which can be treated with the help of the Poisson summation formula. The general problem he could only solve partially, but he was able to continue analytically to the region $\operatorname{Re} s > 0$ with techniques from operator theory [**19**]. The discrete spectrum lies in the interval $-\frac{1}{4} < \lambda \leqslant 0$ and the associated eigenfunctions are residues of $E(z, s)$.

The problem was also considered by Selberg, who solved it completely [**21**]. For his proof, at least for one of them, the essential tool for the analytic continuation is provided by inequalities for the coefficients in (1) when ψ is an Eisenstein series. These are obtained by integration by parts of truncated functions [**15**] or by Fourier analysis [**16**]. Selberg never published a complete proof (cf. [**20** and **22**]) but the proof of the analytic continuation for series of rank one attached to cusp forms given in [**16**] was inspired by his methods. So it contains the same elements, although a little distorted. The proof in [**15**] is perhaps closer to that of Selberg. Since s and $-s$ yield the same eigenvalue the functions $E(z, -s)$ attached to the various cusps must be expressible in terms of $E(z, s)$, and the resulting functional equations are critical to the proof.

But Selberg's purpose in [**21**] went beyond the spectral theory. A function ψ on the upper half-plane may be identified with a function φ on $G = \mathrm{SL}(2, \mathbf{R})$ invariant on the right under $K = \mathrm{SO}(2)$ by setting $\varphi(g) = \psi(g(i))$. If f is a function on G with compact support and bi-invariant under K then

$$\varphi * f = \int_G \varphi(gh) f(h)\, dh$$

is also the lift of a function invariant under Γ. The operators $\varphi \to \varphi * f$ commute with each other and with Δ, and their spectral theory is identical with that of Δ. They are integral operators with easily computed kernel and, if the quotient of G by Γ is compact and the function smooth, even of trace class. The trace is computed by integrating the kernel over the diagonal, and just as for the character of an induced representation is easily expressed as a sum over conjugacy classes of Γ of orbital integrals of f. This is a form of the Selberg trace formula, in this case a simple but nonetheless powerful tool. If the quotient is not compact the operators are no longer of trace class, but their restriction to the space of cusp forms is. It is still possible with the help of the Eisenstein series to obtain a formula for the trace of the restriction, but the analysis is substantially more difficult and the result far more complicated [**21**].

As an application Selberg evaluated in closed form the trace of the Hecke operators acting on holomorphic forms of a given weight and level, a problem treated at about the same time by Eichler [**8**] with the help of a Lefschetz formula, at least for weight two. For this application one must consider not functions on G/K, which is the upper half-plane, but sections of a bundle defined by K, in other words functions on G transforming on the right according to a certain finite-dimensional representation of K and invariant on the left under Γ. Indeed at the time of writing of [**21**] a number of developments (cf. [**13** and **14**]) were making it clear that the proper setting for the theory of automorphic forms was a reductive group G and an arithmetic subgroup Γ, and that many aspects of it were nothing but a study of the infinite-dimensional representation of G on $L^2(\Gamma \backslash G)$. The origin of these developments is generally felt to be the 1952 paper of Gelfand-Fomin [**10**], in which representation-theoretic methods were introduced into the study of geodesic flows.

The general problems were considered in the addresses of both Gelfand [**11**] and Selberg [**22**] in Stockholm. Selberg works with an arbitrary group, although he confines himself to K-invariant functions. He poses the problem of analytic continuation of the Eisenstein series in general, sketches very clearly his proof in the rank one case, and draws attention to some special series in several variables whose analytic continuation can be effected by means essentially classical. In addition he states that he can treat all series for the pair $\Gamma = \mathrm{SL}(n, \mathbf{Z})$, $G = \mathrm{SL}(n, \mathbf{R})$, but no indications of proofs have ever appeared. It seems they involve theta series and can only be applied to a limited class of groups. He also emphasizes the importance of developing a trace formula in general, and of applying it to the Hecke operators. Gelfand works with $\Gamma \backslash G$ and stresses the spectral problem, which is now to decompose $L^2(\Gamma \backslash G)$ into a direct integral of irreducible representations. He introduces the fundamental notion of cusp form in general, and states the important theorem, due to himself and Piatetskii-Shapiro, that the representation on the space of cusp forms is a discrete sum of irreducible representations when G is semisimple. He also points out the similarity of the problem with that arising in scattering theory, and it is indeed striking and useful to bear in mind, although the analogy cannot be pushed too far and it has not been very profitable to transport methods from one domain to the other.

The spectral analysis of the quantum-mechanical Hamiltonian H for n interacting particles $X_1, \ldots, X_n$ in d-dimensional space often assumes an intuitively very simple form (see [**12**, §13.2] for a brief description and [**1**] for the complete theory). The bound states correspond, if we overlook the movement of the center of gravity, to the discrete spectrum and are finite in number. More generally if we partition $X_1, \ldots, X_n$ into clusters $S_1, \ldots, S_l$ then the Hamiltonian H_j for the particles in S_j alone will have bound states $X_{1,j}, \ldots, X_{m_j,j}$ which can move with a momentum p_j. For each partition and each choice $X_{k_1,1}, \ldots, X_{k_l,l}$ of bound states there will be a subspace of the total Hilbert space $L^2(\mathbf{R}^{nd})$ on which H acts that is isomorphic to $L^2(\mathbf{R}^{ld})$, the underlying parameters being $p_1, \ldots, p_l$, and H will act on this subspace as

$$\sum_{j=1}^{l} \frac{1}{2m_j} P_j^2 + C,$$

where C is the total energy of the bound states. Thus each partition and each family of bound states yields a piece of the total Hilbert space corresponding to freely and independently moving clusters in these states. The total space is the orthogonal direct sum of the pieces.

The analogue in the theory of Eisenstein series of a partition into clusters is a cuspidal subgroup of G, which is in particular a parabolic subgroup. If $G = \mathrm{GL}(n)$ these are obtained by choosing a basis $\{x_1, \ldots, x_n\}$ of the n-dimensional coordinate space and a partition $S_1, \ldots, S_l$ of the basis. If P_j is the stabilizer of the span of $\bigcup_{1 \leq k \leq j} S_k$ then the parabolic subgroup associated to the basis is $\bigcap_{j=1}^{l} P_j$.

In general if P is a cuspidal subgroup for Γ and if one projects $\Gamma \cap P$ on a Levi factor of P one obtains a pair Θ, M like Γ, G. The Levi factor itself is AM where A is a vector group. A complex character $\chi = \chi(s_1, \ldots, s_l)$ of A depends on l complex parameters and if Φ is a function yielding a discrete part of the spectrum for $\Theta \backslash M$ we can lift the product $\chi \cdot \Phi$ to a function on P. The parameter s_j is the analogue of $\sqrt{-1}\, p_j$ and Φ is the analogue of the family of bound states.

Taking a function F on $G = PK = NAMK$ of the form

$$F(g, s_1, \ldots, s_l) = F(g, s) = F(pk, s) = F(namk, s) = \chi(a) \sum_{j=1}^{n} \Phi_j(m)\Psi_j(k) \tag{2}$$

we form the Eisenstein series

$$E(g, s) = \sum_{\Gamma \cap P \backslash \Gamma} F(\gamma g, s). \tag{3}$$

It converges in a tube over a cone, but not over the point needed for the spectral analysis, and if the emphasis is on Eisenstein series as in **[22]** the problem is to show that these functions can be analytically continued as meromorphic functions to all of $\mathbf{C}^l$, and that they satisfy functional equations. If the emphasis is on the spectral decomposition of $L^2(\Gamma\backslash G)$ it must be shown as well how they yield the spectral decomposition of $L^2(\Gamma\backslash G)$. So far it has not been possible to solve the first problem without at the same time solving the second. They were both solved in **[16]**. Selberg has recently indicated to me that he had an idea for effecting analytic continuation without reference to a spectral decomposition but with the help of Fredholm theory. However he has not developed it. It would be worthwhile to do so.

The argument of **[16]** requires some geometrical assumptions on Γ. The ones used are adequate to arithmetic groups, and indeed based on their reduction theory, and to Fuchsian groups of the first kind. They allow one to introduce the constant terms

$$\int_{\Gamma \cap N \backslash N} \varphi(ng)\, dn,$$

to define the space of cusp forms, as consisting of those functions whose constant term is zero for all cuspidal subgroups but G itself, to control the behavior of eigenfunctions by means of their constant terms, an important analytic tool, and in particular to establish the theorem of Gelfand-Piatetskii-Shapiro.

Then if A is of dimension one, so that $l = 1$ and the series depend on a single complex variable, and if the functions Φ_j are taken to be cusp forms the proof of the analytic continuation and the functional equation proceeds pretty much as for subgroups of $\mathrm{SL}(2, \mathbf{R})$. If the dimension of A is greater than one but the Φ_j continue to be cusp forms, then a truncation argument and a partial summation to reduce to the one-dimensional case yield the result. The argument to this point is also presented in **[15]**.

The method used in **[16]** to deal with the general Eisenstein series is to show that it can be obtained from a series associated to a cusp form by taking a succession of residues, reducing thereby the number of variables at each stage by one. It is related to the fact that in two-particle scattering problems the bound states appear at poles of the scattering matrix. The central difficulty is to convince oneself that all Eisenstein series are obtained in this way. The analytic continuation is then immediate, and the functional equations and spectral decomposition are obtained in the course of the argument.

Its basic nature is easily described. If a function φ on $N\backslash G$ has compact support then

$$(4) \qquad \theta(g) = \sum_{\Gamma\cap N\backslash\Gamma} \varphi(\gamma g)$$

is square-integrable on $\Gamma\backslash G$ and if φ can be represented as

$$\varphi(amk) = \frac{1}{(2\pi)^l}\int_{\operatorname{Re} s=\sigma} \chi(a, s)\alpha(s)\,|ds_1|\cdots|ds_l|\sum_j \Phi_j(m)\Psi_j(k),$$

where $s = (s_1,\ldots,s_l)$ and the Φ_i are cusp forms then

$$(5) \qquad \theta(g) = \frac{1}{(2\pi)^l}\int_{\operatorname{Re} s=\sigma} \alpha(s)E(g, s)\,|ds_1|\cdots|ds_l|$$

and there is a fairly simple expression for the L^2-norm of θ in terms of α, which is an entire function, and certain auxiliary functions. In the simplest cases it is of the form

$$(6) \qquad \int_{\Gamma\backslash G}|\theta(g)|^2\,dg = \frac{1}{(2\pi)^l}\int_{\operatorname{Re} s=\sigma}\sum_{\omega\in\Omega} m(\omega, s)\alpha(s)\bar{\alpha}(-\omega\bar{s})\,|ds|\,.$$

The group Ω is a finite group of real linear transformations, a Weyl group, and the functions $m(\omega, s)$, which appear in the constant term of the Eisenstein series, satisfy

(i) $m(1, s) = 1$,
(ii) $\overline{m}(\omega, s) = m(\omega^{-1}, -\omega\bar{s})$,
(iii) $m(\omega_1\omega_2, s) = m(\omega_1, \omega_2 s)m(\omega_2, s)$.

In particular $|m(\omega, s)| = 1$ if s is purely imaginary.

The problem is in essence to find a decomposition for the space spanned by the functions θ. If σ in the formula (6) were 0 then

$$\int_{\Gamma\backslash G}|\theta(g)|^2\,dg = |\Omega|\int_{\operatorname{Re} s=0}|\beta(s)|^2\,|ds|$$

where $\beta = \Pi\alpha$ is defined by

$$\beta(s) = \frac{1}{|\Omega|}\sum_{\omega\in\Omega} m(\omega^{-1}, \omega s)\alpha(\omega s).$$

The operator Π is the orthogonal projection of the space of square-integrable functions on $\operatorname{Re} s = 0$ onto the space of β satisfying

$$\beta(\omega s) = m(\omega, s)\beta(s)$$

for all ω and s with $\operatorname{Re} s = 0$. Thus an obvious density argument yields an isomorphism of the space spanned by the θ with a simple L^2-space, and it has been so constructed that the operators of interest become multiplication by functions of s.

Unfortunately σ is usually not 0. The procedure in general is to deform the contour to $\sigma = 0$, thereby picking up residual integrals of dimension $l - 1$, the poles of the functions $m(\omega, s)$ being the poles of Eisenstein series. If α is chosen to vanish along these poles these residues do not appear. Since this restriction does not affect density in the space of square-integrable functions on $\operatorname{Re} s = 0$, the l-dimensional spectrum is as before. For an arbitrary α the square of the norm of the projection of θ on the complement of this spectrum is given by the residual integral. But then the process can be iterated until one arrives at the discrete spectrum and is done.

There are difficulties. The analogues of the functions $m(\omega, s)$ may have poles of high order; they may have poles on the analogues of $\operatorname{Re} s = 0$; and we may be forced into regions in which we can no longer control their rate of growth as $\operatorname{Im} s \to \infty$. So an elaborate induction is required. There is a great deal to be proven at each stage, and to facilitate matters, the notion of Eisenstein system, which supplies the title to the book under review, was defined.

The book is indeed largely an exposition of that part of [**16**] which treats the Eisenstein series associated to general forms and the spectral decomposition. Some find it a useful adjunct to [**16**]; others do not. It must certainly be used with caution, for it is tendentious, the tone occasionally lapsing into truculence.

The first chapter contains a review of results on discrete groups, many with only a tenuous connection with the problems to be treated, and a bizarre survey of previous work on the analytic theory. In particular the reader is misled about the present status of the trace formula and about the role played by adele groups. The first reason for introducing the adele groups into the theory of automorphic forms is the formal and conceptual simplicity they entail. This is particularly true in the theory of Eisenstein series. Moreover the spaces that arise in the adelic theory are finite unions of the spaces $\Gamma\backslash G$ that occur when working with discrete subgroups of Lie groups. So it demands no additional analysis, simply a routine and formal re-interpretation of the results. Osborne and Warner do their readers a disservice by suggesting otherwise.

To confine oneself to adele groups is equivalent to confining oneself to congruence subgroups and it is best to refrain from this until it is appropriate, for the theory of Eisenstein series promises to have applications to the study of the cohomology groups of Γ, and these are of interest for more general classes of discrete groups.

For the trace formula too one hesitates to impose gratuitous restrictions, for it does have geometric applications. However, as appears already in Selberg and as has been confirmed by later applications to Artin and Hasse-Weil L-functions, a principal purpose of the trace formula is to study the Hecke operators, which in general can only be handled adelically. So it is convenient to derive it directly in the adelic context, indeed critical. First of all the trace formula appears as a sum over conjugacy classes, and these are easier to analyze in $G(\mathbf{Q})$ than in $G(\mathbf{Z})$. Secondly both Arthur, who has developed a general trace formula, and Flicker, who has made several interesting applications in low dimensions, exploit devices peculiar to adele groups.

Only the final chapter of the present book refers directly to the trace formula. The authors show, using a device first introduced into the subject by Duflo-Labesse [7], that convolution with a large class of functions yields operators of trace class on the space of cusp forms and, in addition, operators on the total continuous spectrum with continuous kernels, a result due to themselves. The concept of a trace formula implicit in this chapter and in their introduction ignores the experience of the past decade. It differs from that of Arthur, which is highly developed [**2**, **3**], has been applied [**9**], and has led to a body of results of interest in their own right [**4**–**6**]. Incidentally, in Arthur's hands the trace formula has taken a shape somewhat different than anticipated. He introduces directly a truncated kernel, evaluates its trace in two ways, and then deals with the problem of interpreting both sides.

In the second chapter Osborne and Warner devote considerable space to their geometric assumptions on Γ, finally equivalent to those of [**16**], and they point out that is easy, by introducing a compact factor, to construct groups which violate them. The compact factor is a standard device for dealing with cohomology of Γ with coefficients; so there is motivation for extending the theory to these groups, even though it does not appear to be needed for arithmetical purposes, but the authors do not pursue the problem. Chapters 3 and 4 are reviews of material on automorphic forms and Eisenstein series associated to cups forms.

Chapters 5, 6, and 7 are the heart of the book and are an exposition of the induction argument of Chapter 7 of [**16**]. This induction demands the verification of a number of technical conditions at each stage, and a feature of their presentation, which will be useful even to the reader of [**16**], is that they label these conditions, and clarify their logical interdependence. In addition a number of facts, like those of Propositions 5.1, 5.2, 5.7 and Lemma 5.5, which are simply taken for granted or stated without comment in [**16**] are isolated and proved, and this may be a help to the inexperienced reader. On the other hand the global structure of the induction is obscured. So it may be worthwhile to close the review with a technical discussion of the proofs, in an attempt to provide a guide to these three chapters and to the last chapter of [**16**] as well.

The pair $(\Gamma, G) = (\mathrm{GL}(n_1, \mathbf{Z}) \times \cdots \times \mathrm{GL}(n_r, \mathbf{Z}),\ \mathrm{GL}(n_1, \mathbf{R}) \times \cdots \times \mathrm{GL}(n_r, \mathbf{R}))$ is typical. A conjugacy class of cuspidal subgroups is determined by partitions $\Pi_i = \{S_i^1, \ldots, S_i^{l_i}\}$ of n_i, $1 \leq i \leq r$, and the Levi factor is then isomorphic to $\prod_{i=1}^r \prod_{j=1}^{l_i} \mathrm{GL}(n_i^j, \mathbf{R})$ if $n_i^j = |S_i^j|$. So these pairs are sufficiently general to permit induction.

Two conjugacy classes of cuspidal subgroups are associate if for all i the partition Π_i' is obtained from Π_i by a permutation of $\{1, \ldots, n_i\}$. In contrast to scattering theory there is here an easy initial decomposition of $L^2(\Gamma\backslash G)$ into a direct sum of spaces $L(\mathcal{P})$. The space $L(\mathcal{P})$ is the closed span of the functions θ introduced above as P varies over $\mathcal{P}$. It is the space $L(\mathcal{P})$ we need to decompose.

Let $\mathcal{P}' \succ \mathcal{P}$ mean that some family of partitions defining $\mathcal{P}$ is finer than one defining $\mathcal{P}'$ and let $r(\mathcal{P})$ be the rank of the P in $\mathcal{P}$, namely $\sum_i l_i$. The decomposition of $L(\mathcal{P})$ takes the form

$$L(\mathcal{P}) = \sum_{\mathcal{P}' \succ \mathcal{P}} L(\mathcal{P}', \mathcal{P})$$

where $L(\mathcal{P}', \mathcal{P})$ itself is a direct sum of direct integrals with respect to $r(\mathcal{P}')$-dimensional Lebesgue measure. In particular when $\mathcal{P}' = \{G\}$ the rank

$r(\mathcal{P}')$ is a minimum and

$$L(\{G\}, \mathcal{P}) = \oplus V \otimes L^2(\mathbf{R}^r)$$

where V are subspaces of functions on $\Gamma\backslash G$ square-integrable modulo the centre of G and G acts irreducibly on V. The action on $L^2(\mathbf{R}^r)$ is given by $g = (g_1,\ldots,g_r)\colon f(x_1,\ldots,x_r) \to \Pi_{j=1}^r(\det g)^{ix_j}f(x_1,\ldots,x_r)$. Thus $V \otimes L^2(\mathbf{R}^r) = \int V \otimes \chi(ix_1,\ldots,ix_r)\,dx_1 \cdots dx_r$. The sum runs over all such V modulo the equivalence $V \cong V \otimes \chi(ix_1,\ldots,ix_r)$, $x_1,\ldots,x_r \in \mathbf{R}$.

It is important that the K-finite functions in these spaces can all be expressed as linear combinations of residues of Eisenstein series associated to cusp forms on Levi factors of parabolic subgroups in $\mathcal{P}$. Such a residue is obtained by choosing the parabolic subgroup P and I collections Φ_j^i, Ψ_j^i, $1 \leqslant j \leqslant n_i$, of functions on $\Theta\backslash M$ and K respectively, where each Φ_j^i is a cusp form, building the functions $F_i(g, s)$, then choosing polynomials $a_i(s)$, and finally taking the $(l-r)$-fold residue of $\Sigma_{i=1}^I a_i(s)E_i(g, s)$ with respect to $l - r$ linear functions on $\mathbf{C}^l$.

The analytic continuation of all Eisenstein series is immediate. Consider that defined by (3) in which for convenience we replace P by P'. The functions Φ_j occurring in (2) will then be finite linear combinations of functions in $L^2(\Theta'\backslash M')$ transforming according to an irreducible representation of M'. Using the spectral decomposition we may even suppose that each function $a'm' \to \Phi_j(m')$ lies in some V, with $V \otimes L^2(\mathbf{R}^{l'}) \subseteq L(\{M'\}, \mathcal{P}_{M'})$ and finally that it is a residue of some $\Sigma\, a_i(s)E_i'(g, s)$, the E_i' being Eisenstein series for M' attached to cusp forms on the Levi factor M of a cuspidal subgroup P_M in M'. But P_MN' is then a cuspidal subgroup P of G with Levi factor M and the Eisenstein series attached to the Φ_j is an $(l-l')$-fold residue of $\Sigma\, a_i(s)E_i(s)$, E_i being defined by the same collection as E_i' but as an Eisenstein series on G attached to P. Since $\Sigma\, a_i(s)E_i(s)$ is meromorphic in $\mathbf{C}^{l'}$ the residue is meromorphic on $\mathbf{C}^l$. This gives the analytic continuation and, if one likes, the functional equations as well. In fact this part of the argument must be incorporated into the inductive construction, because as one peels off $L(\mathcal{P}, \mathcal{P})$, $L(\mathcal{P}', \mathcal{P})$, $r(\mathcal{P}') = r(\mathcal{P}) - 1$, $L(\mathcal{P}', \mathcal{P})$, $r(\mathcal{P}') = r(\mathcal{P}) - 2$, and so on, successively from $L(\mathcal{P})$ one must use the analytically continued Eisenstein series to decompose them as direct integrals.

The induction demands a deformation of contours in complex spaces of the form $\operatorname{Re} s = \sigma$, $s = (s_1,\ldots,s_l)$. The first point to check is that this does not force one to contend with infinitely many residues. This is Proposition 5.3 of Osborne-Warner (Lemma 7.2 of [**16**]). The proof requires Lemma 7.3 of [**16**], which does not seem obvious to me but which Osborne-Warner insert as an observation, with no proof and no comment but a page number in [**16**].

The $(l - l')$-fold residues arising from the integrals (5) will be of the form

$$\frac{1}{(2\pi)^{l'}}\int \alpha'(s)E'(g, s')\,|d's| \tag{7}$$

where s' lies in an l-dimensional space $X \subseteq \mathbf{C}^l$. There are many of them and considerable redundancy can occur, in the sense that the eigenfunctions

$E'(g, s')$ parametrized by one space may be the same as those parametrized by another. The purpose of Proposition 5.4 (Lemma 7.4 of **[16]**) is to control the redundancy. It also yields the functional equations immediately, but the authors do not point this out clearly. Rather they devote a separate Chapter 6 to them, burying a simple fact in a welter of notation. Proposition 5.6 (Corollary to Lemma 7.4 in **[16]**) guarantees in essence that the residues one obtains are eigenfunctions.

The technical device used to overcome the lack of information on the growth of the functions $E'(g, s')$ in (7) on the sets $\operatorname{Re} s' = \sigma'$ is the spectral theory of an operator analogous to the Hamiltonian. It is used in Propositions 5.8 and 5.9 (Lemmas 7.5 and 7.6 of **[16]**) to construct the spaces $L(\mathfrak{P}', \mathfrak{P})$. Their structure is manifest, for at this point one has the analytic continuation of the relevant Eisenstein series, although one has to take Proposition 5.11 (Corollary to Lemma 7.6), which guarantees that they are analytic on the unitary axis, into account. Thus one has the spectral decomposition, but Osborne and Warner wait until Chapter 7, which seems nonetheless to be clearly written, to notice it.

However all this presupposes the successful construction of Eisenstein systems at each stage, and the final struggle comes in proving Theorem 5.12 (Theorem 7.7 ($=$ 7.1) of **[16]**). After the first stage the subspaces X of formula (7) intersect, so that at the following stages there may be several residues attached to the same space X but to different σ'. Thus it is necessary to choose a definite σ_0' and deform all contours to $\operatorname{Re} s' = \sigma_0'$, thereby introducing residues of one dimension less, which have to be set aside momentarily but taken into account at the next step. It is difficult to juggle all these spaces and to ensure that none without the properties essential to the induction insinuate themselves. The argument of **[16]** is compressed into ten pages, but Osborne and Warner wisely take fifty-four, which include however Lemma 7.1 of **[16]**. The arguments are similar but not identical and involve delicate geometric considerations, on which everything hangs, as by a thread.

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CR submanifolds of Kaehlerian and Sasakian manifolds, by Kentaro Yano and Masahiro Kon, Progress in Mathematics, Vol. 30, Birkhauser, Cambridge, Mass., 1982, x + 208 pp., $17.50. ISBN 3-7643-3119-4

The theory of submanifolds of Kaehlerian manifolds is one of the important branches of differential geometry. It began as a separate area of study in the 19th century with the investigation of projective varieties in a complex projective m-space $\mathbf{CP}^m$. It was J. A. Schouten and D. van Dantzig [**10, 11**] who, in 1930, first tried to transfer results in differential geometry of Riemannian manifolds to complex manifolds. In their papers there appeared a Hermitian space with the so-called symmetric unitary connection. The space with the same connection was also found independently by E. Kähler [**8**], and such a space is now called a Kaehlerian manifold. Since then, Kaehlerian manifolds have been studied extensively. Many important results have been obtained.

The study of complex submanifolds of Kaehlerian manifolds from a differential geometric point of view (that is, with emphasis on the Riemannian metric) was initiated by E. Calabi and others more than 30 years ago. Such a theory has become a very active branch of modern differential geometry in the last two decades. In particular, many important results on complex submanifolds in complex-space-forms have been obtained.