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INVARIANT THEORY OF G_2

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Introduction. Let V denote $\mathbf{C}^n$, and let $G \subseteq \mathrm{SL}(V)$ be a classical subgroup. Then Classical Invariant Theory (CIT) describes the generators and relations of the algebra of invariant polynomial functions $\mathbf{C}[mV]^G$, where $m \in \mathbf{Z}^+$ and mV denotes the direct sum of m copies of V. Using the symbolic method (see [**7**]), one can then obtain a handle on the invariants of arbitrary representations of G. These classical methods and results have been very useful in many areas of mathematics.

Let G be a connected, simple, and simply connected complex algebraic group. Then G is classical except when $G = \mathrm{Spin}_n$, $n \geq 7$, or in case G is an exceptional group G_2, F_4, E_6, E_7, or E_8. It would be useful to have an analogue of CIT for nonclassical G. We have succeeded in establishing an analogue for G_2 (described below). We also have a conjectured analogue for Spin_7, but a complete proof requires a computation we are as yet unable to perform.

The Cayley algebra, G_2, and the Main Theorem. Let Cay denote the usual (complex) Cayley algebra (see [**3**]). Then Cay is a nonassociative, noncommutative algebra of dimension 8 over $\mathbf{C}$. Let Cay$'$ denote the (7-dimensional) span of all commutators of elements of Cay. Let $\mathrm{tr}: \mathrm{Cay} \to \mathbf{C}$ denote the linear map with kernel Cay$'$ which sends $1 \in \mathrm{Cay}$ to $1 \in \mathbf{C}$. Define $\bar{x} = -x + 2\,\mathrm{tr}(x) \cdot 1$, $x \in \mathrm{Cay}$. Then $x \mapsto \bar{x}$ is an involution such that $x\bar{x} = n(x) \cdot 1 \in \mathbf{C} \cdot 1$ for all $x \in \mathrm{Cay}$. Moreover,

$$(1) \qquad x(xy) = x^2 y; \quad (yx)x = yx^2, \qquad x, y \in \mathrm{Cay}.$$

$$(2) \qquad x^2 - 2\,\mathrm{tr}(x)x + n(x) \cdot 1 = 0, \qquad x \in \mathrm{Cay}.$$

(3) $\qquad x \mapsto n(x)$ is a nondegenerate quadratic form on Cay.

The identities in (1), called the alternative laws, are a weak form of associativity. Equation (2) is called the standard quadratic identity.

G_2 is the group of algebra automorphisms of Cay. Thus G_2 acts trivially on $\mathbf{C} \cdot 1$ and faithfully (and orthogonally) on Cay$'$. From now on, let G denote G_2 and let V denote Cay$'$. By (3), V is G-isomorphic to its dual V^*.

The following is our main result.

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THEOREM 4. *Let $m \in \mathbf{Z}^+$, and let x_j denote a typical element in the jth copy of V in mV, $1 \le j \le m$.*

(4.1) $\mathbf{C}[mV]^G$ *is generated by elements* $\mathrm{tr}(x_{i_1}(x_{i_2}\cdots x_{i_r})\cdots)$, $r \le 4$.

(4.2) *The relations of these generators are consequences of identities* (1) *and* (2). *Moreover, the relations are generated by ones of degrees* 6, 7, *and* 8 *in the* x_j.

In (4.1) the elements of $V = \mathrm{Cay}'$ are multiplied in Cay; the traces of such products are clearly G-invariant. The relations of (4.2) are obtained by replacing x and y in (1) and (2) by products of elements of Cay′, multiplying the resulting equations by other products, and then taking traces.

Our results are analogous to those of Kostant-Procesi-Rasmyslev (see [**2**]) for the adjoint representation of SL_n. In their case (1) is replaced by associativity, and (2) by the Cayley-Hamilton identity.

As outlined below, we determined generators and relations for $\mathbf{C}[mV]^G$ using techniques of invariant theory and commutative algebra. We then showed, a posteriori, that the generators and relations are as in (4.1) and (4.2). J. Ferrar has informed us that he also has a proof of (4.1).

Generators. We sketch a proof of (4.1): Let $x_1, \ldots, x_m$ be as in the Theorem. Set

$$\alpha_{ij} = -\,\mathrm{tr}(x_i x_j), \qquad 1 \le i, j \le m, \tag{5.1}$$

$$\beta_{ijk} = -\,\mathrm{tr}(x_i(x_j x_k)), \qquad 1 \le i,\ j,\ k \le m, \tag{5.2}$$

$$\gamma_{ijkl} = \text{skew } \mathrm{tr}(x_i(x_j(x_k x_l))), \qquad 1 \le i,\ j,\ k, l \le m, \tag{5.3}$$

where in (5.3) we skew symmetrize in the indices. The invariant α_{ij} is symmetric in its indices, while β_{ijk} and γ_{ijkl} are skew symmetric in theirs (hence are zero if the same index appears twice).

Let ω denote a nonzero element of $(\bigwedge^3 V)^G$ corresponding to the β type invariants. One can show that wedge multiplication by ω gives an isomorphism of $\bigwedge^2 V$ with $\bigwedge^5 V$, and it follows that generators of $\mathbf{C}[mV]^G$ can be obtained by polarization from those in the case $m = 4$. A theorem of Weyl [**7**, p. 154] says that, when $m = 4$, it suffices to consider generators whose degree d in the fourth copy of V is at most 1. These generators correspond to invariants in $\mathbf{C}[3V]$ (if $d = 0$), and copies of $V^* = V$ in $\mathbf{C}[3V]$ (if $d = 1$). Using [**4**] and [**5**] one can show that $\mathbf{C}[3V]^G$ is generated by α and β type invariants, and that the covariants in $\mathbf{C}[3V]$ corresponding to the representation V form a free $\mathbf{C}[3V]^G$-module with three generators in degree 1, three in degree 2, and one in degree 3. The degree 3 generator corresponds to an invariant of type γ; the other generators give nothing new. Thus $\mathbf{C}[mV]^G$ is generated by invariants of types α, β, and γ; establishing (4.1).

Relations (Proof of (4.2)). Since $V \simeq V^*$, we can just as well consider computing the G-invariants of the symmetric algebra $S^*(mV)$. Note that GL_m acts naturally on $S^*(mV)^G \simeq S^*(V \otimes \mathbf{C}^m)^G$.

Let ϕ_i denote the standard representation of GL_m on $\bigwedge^i \mathbf{C}^m$ (so $\phi_i = 0$ if $i > m$). If $a_1, \ldots, a_k \in \mathbf{Z}^+$, let $\phi = \phi_1^{a_1} \cdots \phi_k^{a_k}$ denote the highest weight component of $S^{a_1}(\phi_1) \otimes \cdots \otimes S^{a_k}(\phi_k)$. If $a_k > 0$, we say that ϕ has height k.

The generators of $S^*(V \otimes \mathbf{C}^m)^G$ of (5.1), (5.2), and (5.3) transform by the representations ϕ_1^2, ϕ_3, and ϕ_4, respectively. Thus there is a GL_m-equivariant surjection π from $R = S^*(\phi_1^2 + \phi_3 + \phi_4)$ to $S = S^*(V \otimes \mathbf{C}^m)^G$, and $I = \operatorname{Ker} \pi$ is GL_m-invariant and homogeneous (grade R and S in the obvious way). Comparing R and S in degrees ≤ 8, one finds the following elements of I (see explanation below).

(6.1) $\phi_1\phi_5 \subseteq S^2\phi_3$; $\phi_1\phi_5 \subseteq \phi_1^2 \otimes \phi_4$,

(6.2) $\phi_2\phi_5 \subseteq \phi_3 \otimes \phi_4$; $\phi_2\phi_5 \subseteq S^2\phi_1^2 \otimes \phi_3$,

(6.3) $\phi_1\phi_6 \subseteq \phi_3 \otimes \phi_4$,

(6.4) $\phi_8 \subseteq S^2\phi_4$,

(6.5) $\phi_2\phi_6 \subseteq S^2\phi_4$; $\phi_2\phi_6 \subseteq S^2\phi_1^2 \otimes \phi_4$,

(6.6) $\phi_4^2 \subseteq S^2\phi_4$; $\phi_4^2 \subseteq S^4\phi_1^2$; $\phi_4^2 \subseteq \phi_1^2 \otimes S^2\phi_3$.

Each relation consists of a nontrivial "linear combination" of the given representations in R whose image is zero in S. For example, in (6.1), a highest weight vector of the space of relations is $\sigma + \tau$, where

$$\sigma = \beta_{123}\beta_{145} - \beta_{124}\beta_{135} + \beta_{125}\beta_{134}$$

is a highest weight vector of $\phi_1\phi_5 \subseteq S^2\phi_3$, and

$$\tau = \alpha_{11}\gamma_{2345} - \alpha_{12}\gamma_{1345} + \alpha_{13}\gamma_{1245} - \alpha_{14}\gamma_{1235} + \alpha_{15}\gamma_{1234}$$

is a highest weight vector of $\phi_1\phi_5 \subseteq \phi_1^2 \otimes \phi_4$.

It is not difficult to show that relations (6.1)–(6.6) generate I for any m if they generate in the case $m = 6$ (this has a lot to do with the fact that $\dim V = 7!$), so we may assume $m = 6$. Using techniques of [**5**] one can show that S has a regular sequence $f_1, \ldots, f_{28}$ consisting of 18 forms of degree 2 and 10 forms of degree 3. Since S is Cohen-Macaulay (even Gorenstein [**1**, p. 124]) of dimension 28, we find that $S \simeq \mathbf{C}[f_1, \ldots, f_{28}] \otimes S^0$ (as graded $\mathbf{C}[f_1, \ldots, f_{28}]$-module), where $S^0 = S/(f_1, \ldots, f_{28})$ is an artin algebra. Thus the Poincaré series $P(t)$ of S equals $(1-t^2)^{-18}(1-t^3)^{-10}P^0(t)$, where $P^0(t) = \sum_{i=0}^{l} a_i t^i$ is the Poincaré series for S^0. Since S is Gorenstein, $a_i = a_{l-i}$, $0 \leq i \leq l$, and using a result of Stanley [**6**] one can show that $l = 24$. Since S has generators of degree ≤ 4, it follows that I is generated by elements of degree $\leq l + 4 = 28$. Thus we have to show that (6.1)–(6.6) generate I in degrees ≤ 28. This computation was not easy to do, but was made manageable by the GL_6 symmetry and certain estimates arising out of (6.1)–(6.6).

Details are to appear.

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