BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 3, November 1983

ON THE LOCAL LANGLANDS CONJECTURE IN PRIME DIMENSION

BY PHILIP KUTZKO[1] AND ALLEN MOY

Let F be a local field of residual characteristics p. Then it is a conjecture of Langlands [**JL**] that there should be a natural bijection between the set of n-dimensional semisimple representations of the absolute Weil-Deligne group of F and the set of irreducible admissible representations of $\mathrm{GL}_n(F)$. Some cases of this conjecture have been established [**He**, **JL**, **JPS**, **K**, **M**]. Here we announce further progress toward its verification.

To describe our results, we first note that by work of Bernstein and Zelevinsky [**Z**], one may restrict one's attention to irreducible representations of the Weil-Deligne group on the one hand and irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ on the other hand. In this context, the conjecture says there should exist a bijection $\sigma \mapsto \pi(\sigma)$ of the set $\mathcal{A}_n^0(F)$ of equivalence classes of continuous, irreducible n-dimensional complex representations of W_F, the absolute Weil group of F, with the set $\mathcal{A}^0(\mathrm{GL}_n(F))$ of equivalence classes of admissible irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. This bijection should satisfy the following conditions:

(1.01) $\epsilon(\pi(\sigma), \psi) = \epsilon(\sigma, \psi)$ (see [**D**, **GJ**] for definitions),

(1.02) $\pi(\sigma) \otimes \chi \circ \det = \pi(\sigma \otimes \chi)$ for all quasi-characters χ of F^x,

(1.03) $\omega_{\pi(\sigma)} = \det \sigma$, where $\omega_{\pi(\sigma)}$ is the central character of $\pi(\sigma)$.

We note that if $n = 1$, the existence of such a bijection is a restatement of the fundamental theorem of local classified theory [S]; thus when $n \geq 2$, the conjecture under consideration may be thought of as a nonabelian analogue of that theorem.

When $n \geq 2$ the construction of $\pi(\sigma)$ breaks naturally into two steps.

I. *Construction of $\pi(\sigma)$ when σ is induced from a representation of smaller dimension.* This construction is provided when $n = 2$ by decomposing the Weil representation of $\mathrm{SL}_2(F)$ (see [**JL**]). When $n = 3$, it is obtained by global methods [**JPS**]. When $p \nmid n$ then all n-dimensional irreducible representations σ of W_F are monomial, and one may use a representation which induces σ to construct a supercuspidal representation $\pi'(\sigma)$ of $\mathrm{GL}_n(F)$. This was first done by Howe [**Ho**], who conjectured that $\pi'(\sigma)$ satisfied (1.01)–(1.03). Recently, Moy [**M**] showed that a representation $\pi(\sigma)$ satisfying (1.01)–(1.03) may be obtained by a slight modification of Howe's construction and thus verified the Langlands conjecture in case $p \nmid n$ (one needs, however, that $\operatorname{char} F = 0$ in order that the map $\sigma \mapsto \pi(\sigma)$ be bijective).

When $p \mid n$, however, the above approach appears to fail.

Received by the editors June 27, 1983.

1980 *Mathematics Subject Classification*. Primary 12B25, 20G05; Secondary 22E50, 12B15.

[1]Supported in part by NSF Grant #MP575 07481.

II. *Extension of the map $\sigma \to \pi(\sigma)$ to primitive representations σ.* Such an extension was obtained in case $n = p = 2$ by Kutzko in [**K**] and in case $n = p = 3$ by Henniart in [**He**], thus verifying the Langlands conjecture in these cases.

In general, one has reason to expect that step I above may be approached using global methods and that, in particular, such methods will lead to the construction of $\pi(\sigma)$ when σ is induced from a one-dimensional representation on a normal subgroup of W_F. We here announce that, given such a global construction for monomial representations σ of dimension $n = p$, we are able to construct $\pi(\sigma)$ for an arbitrary p-dimensional representation σ. To be precise, we must make the following

ASSUMPTION 1.1. Let K be a local field of residual characteristic p and let E/K be a cyclic extension of degree p. Let θ be a quasi-character of E^x which is not fixed by the galois group, $\Gamma_{E/K}$, of E/K and let $\sigma = \mathrm{Ind}_{W_E \uparrow W_K} \theta$. Then there is an irreducible supercuspidal representation $\pi(\sigma)$ of $\mathrm{GL}_p(K)$ which satisfies (1.01)–(1.03). Furthermore, the map $\sigma \to \pi(\sigma)$ is injective, and if L is a field over which K is galois, then the map $\sigma \to \pi(\sigma)$ commutes with $\Gamma_{K/L}$.

We then prove

THEOREM 1.2. *Suppose Assumption 1.1 holds for all extensions K/F. Then, given any irreducible p-dimensional representation σ of W_F, there exists an irreducible supercuspidal representation $\pi(\sigma)$ of $\mathrm{GL}_p(F)$ satisfying* (1.01)–(1.03).

We proceed as follows (proofs will be provided elsewhere).

DEFINITION 1.3. Call a galois extension K/F a *good* extension of F, either if $\Gamma_{K/F}$ is cyclic and either $[K:F] \mid p-1$ or $[K:F] \mid p+1$, or if $\Gamma_{K/F}$ is a generalized quaternion group and $[K:F] \mid 2(p+1)$.

PROPOSITION 1.4. *Let K/F be a good extension. Then there is a map $\mathrm{lift}_{K/F}$ which maps irreducible supercuspidal representations of $\mathrm{GL}_p(F)$ to irreducible supercuspidal representations of $\mathrm{GL}_p(K)$ and has the following properties:*

(1.4.1) *If π is an irreducible supercuspidal representation of F, then $\mathrm{lift}_{K/F}\pi$ is fixed under $\Gamma_{K/F}$;*

(1.4.2) $\epsilon(\mathrm{lift}_{K/F}\pi, \psi \circ \mathrm{tr}_{K/F}) = [\epsilon(\pi, \psi)]^{[K:F]} \lambda_{K/F}^{-p} \cdot \delta(\pi)$, *where $\lambda_{K/F}$ is the Langlands constant associated to K/F and $\delta(\pi) = \pm 1$;*

(1.4.3) $\mathrm{lift}_{K/F}(\pi \otimes \chi) = \mathrm{lift}_{K/F}(\pi) \otimes \chi \circ N_{K/F}$ *for any quasi-character χ of F^x;*

(1.4.4) *Let π_K be an irreducible supercuspidal representatin of K which is fixed under $\Gamma_{K/F}$ and let $\omega(\pi_K)$ be its central character. Then if ω is a quasi-character of F for which $\omega_K = \omega \circ N_{K/F}$, there is exactly one irreducible supercuspidal representation π of $\mathrm{GL}_p(F)$ for which $\omega(\pi) = \omega$ and $\mathrm{lift}_{K/F}\pi = \pi_K$.*

PROPOSITION 1.5. *Let σ be an irreducible p-dimensional representation of Γ_F. Then there is a good extension K/F, a cyclic extension E/F and a quasi-character θ of E^x such that $\sigma|_{W_K} = \mathrm{Ind}_{W_E \uparrow W_K} \theta$. Furthermore,*

$$\epsilon(\sigma|_{W_K}, \psi_{K/F}) = [\epsilon(\sigma, \psi)]^{[K:F]} \lambda_{K/F}^{-p} \delta(\sigma), \qquad \text{where } \delta(\sigma) = \pm 1.$$

PROPOSITION 1.6. *With notation as above, and assuming* 1.1, *define the representation* $\pi(\sigma)$ *of* $\mathrm{GL}_p(F)$ *by the conditions* $\mathrm{lift}_{K/F}\pi(\sigma) = \pi(\sigma|_{W_K})$, $\omega(\pi(\sigma)) = \det \sigma$. *Then* $\epsilon(\pi(\sigma), \psi) = \xi\epsilon(\sigma, \psi)$, *where* ξ *is a pth-power root of unity.*

PROPOSITION 1.7. $\pi(\sigma)$ *satisfies conditions* (1.01) – (1.03). *If the map* $\sigma \mapsto \pi(\sigma)$ *given by Assumption* 1.1 *has the additional property that it commutes with* $\mathrm{lift}_{K/F}$, *then the map* $\sigma \mapsto \pi(\sigma)$ *constructed above is unique and injective. If all irreducible supercuspidal representations of* $\mathrm{GL}_p(F)$ *may be constructed by induction from open compact-modulo center subgroups* (*see* [**C**]), *then the map* $\sigma \mapsto \pi(\sigma)$ *is a bijection of the set of equivalence classes of irreducible p-dimensional representations of* $W(F)$ *with the set of equivalence classes of irreducible supercuspidal representations of* $\mathrm{GL}_p(F)$ (*see also* [**Ko**]).

REFERENCES

[**C**] H. Carayol, *Représentations cuspidales du groupe linéaire*, preprint, Univ. of Paris VII, 1982.

[**D**] P. Deligne, *Les constantes des equations fonctionelles des fonctions L*, Lecture Notes in Math., vol. 349, Springer-Verlag, 1973, pp. 501–597.

[**GJ**] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Math., vol. 260, Springer-Verlag, 1972.

[**He**] G. Henniart, *La conjecture de Langlands locale pour* GL(3), Inst. Hautes Étude Sci. notes, 1982.

[**Ho**] R. Howe, *Tamely ramified supercuspidal representations of* $\mathrm{GL}_n(F)$, Pacific J. Math. **73** (1977), 437–460.

[**JL**] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer-Verlag, 1970.

[**JPS**] H. Jacquet, I. Piatetski-Shapiro and J. A. Shalika, *Automorphic forms on* GL(3), Ann. of Math. (2) **109** (1979), 169–258.

[**K**] P. Kutzko, *The Langlands conjecture for* GL_2 *of a local field*, Ann. of Math. (2) **112** (1980), 381–412.

[**Ko**] H. Koch, *On the local Langlands conjecture for central division algebras of index p*, Invent. Math. **62** (1980), 243–268.

[**M**] Allen Moy, *Local constants and the tame Langlands correspondence*, Thesis, Univ. of Chicago, 1982.

[**S**] J. P. Serre, *Corps Locaux*, Hermann, Paris, 1968.

[**Z**] A. V. Zelevinsky, *Induced representations of reductive p-adic groups*, Ann. Sci. École Norm. Sup. **13** (1980), 165–210.

MATHEMATICAL SCIENCES DIVISION, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06457