

**THE NEWTON DIAGRAM OF AN ANALYTIC MORPHISM,
 AND APPLICATIONS TO DIFFERENTIABLE FUNCTIONS**

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Consider a system of equations of the form

$$(1) \quad f(x) = A(x) \cdot g(\phi(x)),$$

where $x = (x_1, \dots, x_m)$, $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$ is an analytic mapping, and $A(x)$ is a $p \times q$ matrix of analytic functions. Given $f(x) = (f_1(x), \dots, f_p(x))C^\infty$, we seek C^∞ solutions $g(y) = (g_1(y), \dots, g_q(y))$. There is a necessary condition on the Taylor series of f at each point. Special cases are classical: when $\phi(x) \equiv x$ we have the division theorem of Malgrange [7, Chapter VI], and when $A(x) \equiv I$, the composition problem first studied by Glaeser [5].

We solve the problem in the case that $\phi(x)$ and $A(x)$ are algebraic (or Nash), using a Hilbert-Samuel stratification associated to (1). Our methods, however, go far beyond this case. We present algebraic criteria for solving (1), based on a fundamental relationship between two invariants of an analytic morphism and an associated "Newton diagram". Hironaka's simple but powerful formal division algorithm [3] is exploited systematically. The only results from "differential analysis" used are Whitney's extension theorem [7, Chapter I] and Łojasiewicz's inequality [7, Chapter IV].

Let $\underline{k} = \mathbf{R}$ or \mathbf{C} . (Some of our assertions hold for other fields.) Let M, N be analytic manifolds (over \underline{k}), and $\phi: M \rightarrow N$ an analytic mapping. Let A be a $p \times q$ matrix of analytic functions on M .

For each $a \in M$, let \mathcal{O}_a (respectively, $\hat{\mathcal{O}}_a$) denote the ring of germs of analytic functions at a (respectively, the completion of \mathcal{O}_a in the Krull topology). Let \mathfrak{m}_a be the maximal ideal of $\hat{\mathcal{O}}_a$. In the case $\underline{k} = \mathbf{R}$, let $C^\infty(M)$ denote the algebra of C^∞ functions on M . There is a Taylor series homomorphism $f \mapsto \hat{f}_a$ from $C^\infty(M)^p$ onto $\hat{\mathcal{O}}_a^p$.

The mapping ϕ induces ring homomorphisms $\phi^*: C^\infty(N) \rightarrow C^\infty(M)$, $\phi_a^*: \mathcal{O}_{\phi(a)} \rightarrow \mathcal{O}_a$, and $\hat{\phi}_a^*: \hat{\mathcal{O}}_{\phi(a)} \rightarrow \hat{\mathcal{O}}_a$. Let $\Phi: C^\infty(N)^q \rightarrow C^\infty(M)^p$ denote the module homomorphism over ϕ^* defined by $\Phi(g) = A \cdot (g \circ \phi)$. Let $\hat{\Phi}_a: \hat{\mathcal{O}}_{\phi(a)}^q \rightarrow \hat{\mathcal{O}}_a^p$ denote the analogous module homomorphism over $\hat{\phi}_a^*$.

Let $(\Phi C^\infty(N)^q)^\wedge$ denote the $C^\infty(N)$ -submodule of $C^\infty(M)^p$ consisting of elements which formally belong to the image $\Phi C^\infty(N)^q$ of Φ ; i.e., $(\Phi C^\infty(N)^q)^\wedge = \{f \in C^\infty(M)^p: \text{for all } b \in \phi(M), \text{ there exists } G_b \in \hat{\mathcal{O}}_b \text{ such that } \hat{f}_a = \hat{\Phi}_a(G_b) \text{ for all } a \in \phi^{-1}(b)\}$. Evidently, $(\Phi C^\infty(N)^q)^\wedge$ is closed in the C^∞ topology.

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THEOREM 1. *Suppose that M, N, ϕ and A are algebraic (or Nash). If ϕ is proper, then $\Phi C^\infty(N)^q = (\Phi C^\infty(N)^q)^\wedge$.*

In the following, we use the same notation for a germ at a point and a representative of the germ in a suitable neighborhood.

Let $s \in \mathbb{N}$. Let $M_\phi^s = \{\mathbf{x} = (x^1, \dots, x^s) \in M^s : \phi(x^1) = \dots = \phi(x^s)\}$, and let $\phi : M_\phi^s \rightarrow N$ be the induced morphism.

Invariants of an analytic morphism. Let $\mathbf{a} = (a^1, \dots, a^s) \in M_\phi^s$. Put $b = \phi(\mathbf{a})$. Let $\mathcal{R}_\mathbf{a}$ denote the submodule $\mathcal{R}_\mathbf{a} = \bigcap_{i=1}^s \text{Ker } \hat{\Phi}_{\mathbf{a}^i}$ of $\hat{\mathcal{O}}_b^q$.

By a lemma of Chevalley [4, §II], there is a function $l = l(k, \mathbf{a})$ from \mathbb{N} to itself with the following property: If $G \in \hat{\mathcal{O}}_b^q$ and $\hat{\Phi}_{\mathbf{a}^i}(G) \in \hat{\mathfrak{m}}_{\hat{\mathcal{O}}_b^q}^{l+1} \cdot \hat{\mathcal{O}}_b^q$, $1 \leq i \leq s$, then $G \in \mathcal{R}_\mathbf{a} + \hat{\mathfrak{m}}_b^{k+1} \cdot \hat{\mathcal{O}}_b^q$. We say there is a *uniform Chevalley estimate* if we can choose $l = k(k)$ independent of \mathbf{a} (locally in M_ϕ^s).

Let $H_\mathbf{a}(k)$ denote the *Hilbert-Samuel function* of the $\hat{\mathcal{O}}_b^q$ -module $\hat{\mathcal{O}}_b^q / \mathcal{R}_\mathbf{a}$; i.e.,

$$H_\mathbf{a}(k) = \dim_{\mathbb{k}} \frac{\hat{\mathcal{O}}_b^q}{\mathcal{R}_\mathbf{a} + \hat{\mathfrak{m}}_b^{k+1} \cdot \hat{\mathcal{O}}_b^q}.$$

We say that $H_\mathbf{a}$ is (*analytic Zariski (upper-) semicontinuous*) if, for every irreducible germ of an analytic subset X at a point of M_ϕ^s , there is a germ of a proper analytic subset Y of X , with the following properties:

- (i) $H_\mathbf{a}(k)$ is constant in $X - Y$ for all k .
- (ii) If $\mathbf{a} \in X - Y$ and $\mathbf{a}' \in Y$, then $H_{\mathbf{a}'}(k) \geq H_\mathbf{a}(k)$ for all k , and $H_{\mathbf{a}'}(k) > H_\mathbf{a}(k)$ for some k .

The Newton diagram. Since all our results are local in N , we assume that N is an open subset of \mathbb{k}^n . Following Hironaka [3], we associate to $\mathcal{R}_\mathbf{a}$ a subset $\mathfrak{N}_\mathbf{a}$ of $\mathbb{N}^n \times \{1, \dots, q\}$.

If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, put $|\beta| = \beta_1 + \dots + \beta_n$. We order the $(n+2)$ -tuples $(\beta_1, \dots, \beta_n, j, |\beta|)$, where $(\beta, j) \in \mathbb{N}^n \times \{1, \dots, q\}$, lexicographically from the right. This induces a total ordering of $\mathbb{N}^n \times \{1, \dots, q\}$.

Let $b = (b_1, \dots, b_n) \in N$. We identify $\hat{\mathcal{O}}_b$ with the ring of formal power series $\mathbb{k}[[y - b]] = \mathbb{k}[[y_1 - b_1, \dots, y_n - b_n]]$. Let $G \in \hat{\mathcal{O}}_b^q$, $G = (G_1, \dots, G_q)$. Write $G_j = \sum_{\beta \in \mathbb{N}^n} g_{\beta,j} (y - b)^\beta$, $1 \leq j \leq q$, where $g_{\beta,j} \in \mathbb{k}$ and $(y - b)^\beta$ denotes $(y_1 - b_1)^{\beta_1} \dots (y_n - b_n)^{\beta_n}$. Let $\nu(G)$ denote the smallest (β, j) such that $g_{\beta,j} \neq 0$.

The *Newton diagram* $\mathfrak{N}_\mathbf{a}$ of $\mathcal{R}_\mathbf{a}$ is defined as $\{\nu(G) : G \in \mathcal{R}_\mathbf{a}\}$. Clearly, $\mathfrak{N}_\mathbf{a} + \mathbb{N}^n = \mathfrak{N}_\mathbf{a}$, where addition is defined by $(\beta, j) + \gamma = (\beta + \gamma, j)$, $(\beta, j) \in \mathbb{N}^n \times \{1, \dots, q\}$, $\gamma \in \mathbb{N}^n$.

REMARKS. (i) There is a smallest finite subset $\mathfrak{A}_\mathbf{a}$ of $\mathfrak{N}_\mathbf{a}$ such that $\mathfrak{N}_\mathbf{a} = \mathfrak{A}_\mathbf{a} + \mathbb{N}^n$.

(ii) $H_\mathbf{a}(k)$ is the number of elements $(\beta, j) \in \mathbb{N}^n \times \{1, \dots, q\}$ such that $|\beta| \leq k$ and $(\beta, j) \notin \mathfrak{N}_\mathbf{a}$.

THEOREM 2. *The following conditions are equivalent:*

- (2.1) *There is a uniform Chevalley estimate.*
- (2.2) *The Hilbert-Samuel function $H_\mathbf{a}$ is Zariski semicontinuous.*

(2.3) Each point of M_ϕ^s admits a neighborhood U and a filtration $U = X_0 \supset X_1 \supset \dots \supset X_l = \emptyset$ by closed analytic subsets of U , such that the Newton diagram \mathfrak{N}_a is constant on $X_i - X_{i+1}$ for each i .

Suppose $k = \mathbf{R}$. If ϕ is proper, then (locally in N) there is a bound s on the number of connected components of a fiber $\phi^{-1}(b)$. With this s , we prove:

THEOREM 3. *Suppose ϕ is proper. Then each of the conditions of Theorem 2 implies $\Phi C^\infty(N)^q = (\Phi C^\infty(N)^q)$.*

IDEA OF THE PROOF. It follows from Theorem 2 that there is a locally finite partition $\{X_i\}$ of M_ϕ^s such that, for each i :

- (i) X_i is a connected smooth semianalytic subset of M_ϕ^s .
- (ii) $\overline{X}_i - X_i \subset Y_{i-1}$, where $Y_i = \bigcup_{j \leq i} X_j$.
- (iii) For all $a \in X_i - \phi^{-1}(\phi(Y_{i-1}))$, $\mathcal{R}_a = \bigcap_{a \in \phi^{-1}(\phi(a))} \text{Ker } \hat{\Phi}_a$.
- (iv) \mathfrak{N}_a is constant on X_i .

Let $f \in (\Phi C^\infty(N)^q)$. By induction on i , we assume f is flat on $\phi^{-1}(\phi(Y_{i-1}))$. Let $a = (a^1, \dots, a^s) \in X_i - \phi^{-1}(\phi(Y_{i-1}))$, $b = \phi(a)$. By (iii), (iv) and Hironaka's formal division theorem [3], there is a unique $G = G_b \in \hat{\mathcal{O}}_b^q$ such that $\hat{f}_a = \hat{\Phi}_a(G_b)$ for all $a \in \phi^{-1}(b)$, and $g_{\beta,j} = 0$ for all $(\beta, j) \in \mathfrak{N}_a$. Then the G_b are induced by a q -tuple of C^∞ functions which are flat on $\phi(Y_{i-1})$ (cf. [2]).

THEOREM 4. *The conditions of Theorem 2 are satisfied in each of the following cases:*

- (4.1) M, N, ϕ and A are algebraic.
- (4.2) $\phi = \text{identity}$.
- (4.3) $A = I$ and ϕ is regular; i.e., for each $a \in M$, the Krull dimension of $\mathcal{O}_{\phi(a)}/\text{Ker } \phi_a^*$ equals the generic rank of ϕ near a .
- (4.4) ϕ is finite.

REMARKS. (i) The algebraic hypothesis in (4.1) is essential only to the following point in our proof: If $a \in M$, $b = \phi(a)$, then any $G \in \hat{\mathcal{O}}_b^q$ such that $\hat{\Phi}_a(G) = 0$ can be approximated to any order by an algebraic solution. Writing $y = \phi(x)$ in local coordinates, this amounts to considering the system of equations $A(x) \cdot g(y) = \sum_{i=1}^n h_i(x, y)(y_i - \phi_i(x))$, and finding an algebraic approximation $g(y)$, $h_i(x, y)$ to a given formal solution. Since the equations are linear in the h_i , this special case of "Artin approximation with respect to nested subrings" follows from Artin's theorem [1].

(ii) In the classical coherent case (4.2), the uniform Chevalley estimate is equivalent to a "uniform Artin-Rees lemma" (cf. [9]), and the conclusion of Theorem 3 is Malgrange's theorem on ideals generated by analytic functions. The uniform Chevalley estimate (2.1) in case (4.3) can be obtained using techniques of [9] (as Tougeron showed us); Theorem 3 in this case gives the composition theorem of [2]. In case (4.4), condition (2.2) follows from the finite coherence theorem of Grauert and Remmert [6], and Theorem 3 recovers a result of Merrien [8].

Detailed proofs of our theorems will appear.

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