

4. A. Grothendieck and J.-P. Serre, *Techniques de construction en geometrie analytique*, Seminaire H. Cartan, Paris, 1960/61. Exp. 7–12 and the Appendix by Serre.
5. A. H. M. Hoare, A. Karrass and D. Solitar, *Subgroups of NEC groups*, Comm. Pure Appl. Math. **26** (1973), 731–744.
6. A. Hurwitz, *Algebraische Gebilde mit eindeutigen Transformationen in sich*, Math. Ann. **41** (1893), 403–442.
7. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*. Ergeb. Math. Grenzgeb. vol. 89, Springer-Verlag, Berlin, 1977.
8. A. M. Macbeath, *Groups of hyperbolic crystallography*, Math. Proc. Cambridge Philos. Soc. **81** (1975), 699–712.
9. _____, *The classification of noneuclidean plane crystallographic groups*, Canad. Math. J. **6** (1976), 1192–1205.
10. W. Magnus, *Noneuclidean tessalations and their groups*, Academic Press, New York, (1974).
11. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Wiley, New York, 1966.
12. A. Marden, I. Richards and B. Rodin, *Analytic self-mappings of Riemann surfaces*, J. D'Analyse **18** (1967), 197–225.
13. M. C. Wilke, *On noneuclidean crystallographic groups*, Math. Z. **91** (1966), 87–102.
14. H. Zieschang, E. Vogt and H.-D. Coldewey, *Flächen und ebene diskontinuierliche Gruppen*, Lecture Notes in Math., vol. 122, Springer-Verlag, Berlin and New York, 1970.
15. B. Zimmerman, *Eine Verallgemeinerung der Formel von Riemann-Hurwitz*, Math. Ann. **229** (1977), 279–288.

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Mathematical theory of entropy, by Nathaniel F. G. Martin and James W. England, Encyclopedia of Mathematics and its Applications, vol. 12, Addison-Wesley Publishing Company, Reading, Mass., 1981, xxi + 257 pp., \$29.50. ISBN 0-2011-3511-6

Topics in ergodic theory, by William Parry, Cambridge Tracts in Mathematics, vol. 75, Cambridge Univ. Press, Cambridge, England, 1981, x + 110 pp., \$23.95. ISBN 0-5213-3986-3

An introduction to ergodic theory, by Peter Walters, Graduate Texts in Math., vol. 79, Springer-Verlag, Berlin and New York, 1982, ix + 250 pp., \$28.00. ISBN 0-3879-0599-5

Ergodic theory is concerned with the action of a transformation T or group of transformations G on a space X . The space X usually has some measure-theoretic, topological, or smooth structure which T or G preserves. Typical of the kinds of questions asked is the early recurrence theorem of Poincaré, which states that if T preserves a finite measure then almost every point in any set of positive measure must return to the set under the action of T . The much deeper convergence theorems of von Neumann and Birkhoff triggered considerable

interest in the subject in the 1930s and 1940s. These convergence theorems state that if T preserves a finite measure m then the averages

$$\frac{1}{n} [f(x) + f(Tx) + f(T^2x) + \cdots + f(T^{n-1}x)]$$

will converge in L^2 -norm if f is square-integrable (von Neumann) and almost everywhere if f is integrable (Birkhoff) and are known, respectively, as the mean and individual (or pointwise) ergodic theorems. The von Neumann proof used the idea that T defines an operator U on functions by the formula

$$Uf(x) = f(Tx)$$

and was subsequently generalized to obtain convergence theorems for operator averages in more and more general Banach space settings, culminating in the Yoshida-Kakutani theorem giving necessary and sufficient conditions for convergence in an arbitrary Banach space. The Birkhoff proof involved showing that the maximum, in n , of the averages could not be too big except on a set of small measure, a result that became known as the maximal lemma. Many proofs and generalizations were obtained for the maximal lemma, culminating in the short, elegant result of Garcia, which can be used to obtain the general ratio ergodic theorem of Chacon and Ornstein.

The major emphasis in contemporary ergodic theory, isomorphism theory, traces its origins to an early result of Halmos and von Neumann, who showed in 1942 that the eigenvalues of the unitary operator defined by an irrational rotation form a group, the corresponding eigenspaces are one dimensional and span L^2 , and any measure-preserving transformation whose unitary operator has these same properties must be measure-theoretically isomorphic to translation on a compact group. A measure isomorphism is an invertible measure-preserving map ϕ , from the T space to the S space such that $\phi T = S\phi$. Examples were soon found showing that unitary equivalence did not, in general, imply measure isomorphism, and many concepts, such as ergodicity and weak- and strong-mixing, were shown to be isomorphism invariants. A transformation T is *ergodic* if it has no invariant sets of positive measure, except sets of full measure, which is equivalent to the condition

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} m(T^{-i}A \cap B) = m(A)m(B)$$

for every pair of measurable sets A and B . *Weak-mixing* is obtained by requiring that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} |m(T^{-i}A \cap B) - m(A)m(B)| = 0$$

while *strong-mixing* requires that $\lim_{n \rightarrow \infty} m(T^{-n}A \cap B) = m(A)m(B)$. Irrational rotations are easily seen to be ergodic but not weak-mixing and examples were constructed of transformations that are weak- but not strong-mixing. A complete discussion of these and many other early ideas is contained in the beautiful set of lectures by Halmos, [H]. While many interesting results

were obtained, little further progress was made in finding isomorphism invariants until Kolmogorov showed in 1958 how Shannon's concept of entropy, which had proved so useful in information theory, could be extended to obtain an invariant for measure-preserving transformations.

Kolmogorov defined the entropy of a finite partition P by the formula

$$H(P) = - \sum_{P_i \in P} m(P_i) \log m(P_i)$$

then defined the entropy of T on P by the formula

$$H(T, P) = \lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{i=1}^n T^{-i} P \right)$$

where $\bigvee_{i=1}^n T^{-i} P$ denotes the join (i.e., common refinement) of the partitions $T^{-1}P, T^{-2}P, \dots, T^{-n}P$. Finally Kolmogorov defined the entropy $H(T)$ of T as the supremum of $H(T, P)$ over finite partitions P . This clearly gives an isomorphism invariant for T and in many interesting cases can be easily calculated. In particular, if T is the shift on $X = S^{\mathbb{Z}}$, the space of doubly infinite sequences from a finite set S , and m is the product measure on X determined by a probability distribution $p(s)$ on S , then T is called *the Bernoulli shift determined by $p(s)$* and

$$H(T) = - \sum_{s \in S} p(s) \log p(s).$$

Kolmogorov's entropy idea stimulated a flurry of activity in the 1960s, especially in Russia. Entropies of many transformations of interest were calculated and a wide class of transformations were shown to have completely positive entropy. A transformation has completely positive entropy if $H(T, P)$ is positive for any nontrivial partition P . Transformations with this property are now called *Kolmogorov or K-automorphisms*. The class of K -automorphisms includes Bernoulli shifts, the shifts defined by aperiodic Markov chains, ergodic toral automorphisms, geodesic flows on manifolds of negative curvature, flows on many interesting billiard tables, and many other transformations that arise naturally in the study of dynamical systems. The excellent monograph of Arnold and Avez includes many of these results, [AA]. Much work went into the problem of determining whether entropy was a complete invariant for any class of transformations, in particular, for the class of Bernoulli shifts. Results were obtained by Meshalkin for a small class of Bernoulli shifts, by Adler and Weiss for toral automorphisms, and by Sinai who showed that an ergodic transformation always had a Bernoulli partition of full entropy, that is, a partition P such that $H(T, P) = H(P)$ and $\{T^i P\}$ is an independent sequence.

Ornstein solved the isomorphism problem for Bernoulli shifts in 1969 when he showed that two Bernoulli shifts with the same entropy are isomorphic. His remarkable paper contained many new ideas and was immediately followed by a series of papers by Ornstein and others providing general conditions for a transformation to be isomorphic to a Bernoulli shift and showing that all the previous examples of K -automorphisms were in fact isomorphic to Bernoulli

shifts. Moreover, Ornstein constructed examples of K -automorphisms that are not isomorphic to Bernoulli shifts. These early results are discussed in Ornstein's 1973 Yale lectures, [O]. Much of the recent work in ergodic theory has been concerned with extending the Ornstein theory to more general group actions or with interpreting these ideas in other settings, such as dynamical systems theory and information theory.

Another area of recent activity is the interaction between number theory and ergodic theory, stimulated by Furstenberg's work on Szemerédi's theorem, which states that any set of integers of positive density contains arbitrarily long arithmetic progressions. Furstenberg proved and extended Szemerédi's results by using deep representation theorems that show how general measure-preserving transformations can be built up from translations on compact groups and weak mixing transformations. (See [F].) Topological ergodic theory, which is concerned with asymptotic properties of powers of continuous mappings on compact metric spaces, has also been an area of much recent interest, stimulated by entropy concepts, statistical mechanics, and the famous Smale horseshoe, which shows that shifts on sequence spaces naturally arise in the study of differentiable dynamical systems.

So much has been happening in ergodic theory in the past few years that it would be difficult to produce a single book covering all the new concepts. The three books reviewed here are each concerned with only a few of the recent ideas. William Parry's short book concentrates on functional-analytic aspects of ergodic theory, while the book by Nathaniel Martin and James England is mostly concerned with the theory and interpolation of the entropy concept. Peter Walters' book is a substantial revision of his earlier Springer lecture notes and focuses on the topological parts of the theory.

The Walters book is the best of these three books and is recommended as a good place to learn some standard ergodic theory along with some useful topological theory. The ideas are generally carefully presented, many examples are given and the author has included extensive notes guiding the reader to further results. The lack of exercises, however, make the book less suitable as a text. More detailed comments are given in the next three paragraphs.

The first three chapters of the Walters book contain, roughly, the contents of the Halmos book. Walters begins with a careful treatment of the ergodic theorems and useful remarks on their interpretation. He also has a good discussion of mixing properties and the Halmos-von Neumann theory of rotations and includes many examples of transformations. Unfortunately, he has not included a proof of the important ergodic decomposition theorem, in spite of the fact that he makes use of it in later chapters.

Chapter 4 of the Walters book contains an introduction to the Kolmogorov theory of entropy. Elementary parts of the theory are presented in detail, including many results about K -automorphisms. The results of Ornstein and others about the isomorphism theory of Bernoulli shifts are stated without proof. A weakness of the chapter is the author's failure to discuss in detail the important entropy convergence theorem of Shannon-McMillan-Breiman. This theorem (often called the asymptotic equipartition theorem) connects entropy with an estimate of the number of sequences of length n that are likely to occur

in an ergodic process, and is the basis for Shannon information theory as well as the primary reason that entropy is such a useful concept in ergodic theory.

The last half of Walters' book contains an introduction to the ergodic properties of continuous mappings on compact metric spaces. Chapter 5 begins with the elementary ideas connected with the existence of dense orbits and periodic points. A detailed discussion of isometries and expansive maps is given, including the basic result that an expansive map is topologically equivalent to the restriction of the shift on some sequence space to a closed invariant set. Chapter 6 is concerned with the existence and properties of invariant measures for continuous maps. The Adler-Konheim-McAndrew-Bowen concept of topological entropy of a mapping, which gives a rough measure of how orbits are spreading, is discussed in detail in Chapter 7. Chapters 8 and 9 contain detailed results about the connection between topological and measure-theoretic entropy and the author's generalization of Ruelle's concept of pressure. These final two chapters are considerably more difficult than the preceding chapters and would have benefitted from more examples and more discussion of the significance and interpretation of the ideas.

William Parry's short monograph begins with a nice introduction to the origins of ergodic theory in classical mechanics and its later developments. There are chapters on the ergodic theorem, the entropy convergence theorem, mixing concepts, and entropy, and a final chapter of examples. Chapter 1 begins with one of the first ergodic theorems, Weyl's beautiful result that a sequence $\{x_n\}$ is uniformly distributed in the unit interval if and only if $\frac{1}{N} \sum_{n=1}^N \exp(2\pi i k x_n) \rightarrow 0$ for every integer $k \neq 0$, showing how this result is related to theorems about iterations of transformations, decompositions of function spaces, and the existence of invariant measures. Recurrence theorems of Poincaré and Khintchine, the ergodic theorems of von Neumann and Birkhoff and Wiener's dominated ergodic theorem are well presented and supplemented with good exercises: an excellent introduction to the circle of ideas surrounding the ergodic theorem!

In Chapter 2 Parry introduces the concept of information function for countable partitions and then proves the entropy convergence theorem using martingale theorems and a result of Chung. Chapter 3 emphasizes the connection between various mixing properties and properties of the unitary operator U defined by T . Chapter 4 contains a quick sketch of entropy as an isomorphism invariant followed by a careful treatment of the Rohlin-Sinai-Pinsker theory of K -automorphisms. Chapter 5 includes results about changing velocity in a flow to obtain spectral properties and an old example of Kakutani and von Neumann of a transformation that is weak-mixing but not strong-mixing.

The Parry book is well written, the chapter on the ergodic theorem is superb, and the book includes some results that do not appear in other books. The book is very short, however, focusing mostly on old results, and it was disappointing to see such a limited book by one who has contributed so much to ergodic theory. Fortunately, Parry has just published an interesting monograph with Tuncel, containing many new ideas about various kinds of isomorphisms between Markov shifts [PT].

The Martin-England book is intended to be “a self-contained treatment of all the major properties of entropy and its extension, with rather detailed proofs, and... to give an exposition of its uses in those areas of mathematics where it has been applied with some success.” The book includes chapters on probability theory, entropy, information theory, ergodic theory, topological dynamics and statistical mechanics. A good introduction to entropy already exists (Billingsley’s 1965 book, [B]), but there is certainly room for a more up-to-date treatment. While the Martin-England book does contain some useful material, most of the topics treated in detail are done better elsewhere, the notation is often confusing, and some important ideas are given too little attention.

The best part of the Martin-England book is Chapter 2 on the entropy concept and its associated limit theory. More detail is given than by either Walters or Parry, making this part a useful extension to countable and continuous partitions of the treatment in Billingsley. The summary of recent extensions of the Ornstein theory in Chapter 4 is well annotated and hence useful. The discussions of K -automorphisms in Chapter 4, topological entropy in Chapter 5, and random fields in Chapter 6 are also fairly good, although too sketchy. (Walters’ treatment of the first two topics is better and the fine monograph by Kindermann and Snell, [KS], is a better introduction to random fields.)

Other topics discussed in detail by Martin and England include the Rohlin theory of measurable partitions and the Ornstein isomorphism theorem. The decision to make heavy use of the Rohlin theory, a theory designed to classify sub- σ -algebras of Lebesgue sets, is one reason for the notational complexity of the book. For example, the proof of the existence of regular conditional probabilities (Theorem 1.14) involves seven versions of the letter “ P ”. (Shades of Kleene’s famous variations on the letter “ a ”!) Apparently even the authors got confused for the proof is incorrect— $\mu_F(A)$ should be defined to be $P(N_\xi^{-1}(A) \cap F)$ and not $P_\xi(A \cap N_\xi(F))$. It would have been better to omit the Rohlin theory, since the results discussed in the book can be obtained with less notational complexity by more direct means, as done in [B]. The discussion of the Ornstein isomorphism theorem in Chapter 4 is also made more complicated by poor choice of notation. For example, eight variations on the letter “ ξ ” are used and the discussion of the simple concept of “name” on p. 174 is overdone. Their discussion of the Ornstein theorem is essentially only a translation into their notation of the reviewer’s monograph, [S].

The information theory discussion in Chapter 3 focuses on a theorem of Gray, Neuhoff, and Ornstein which shows how block source coding is related to stationary coding. The chapter contains a sketch of some of Shannon’s ideas, including an incorrect proof that capacity for a memoryless channel is attained at a memoryless source, and a statement, without proof, of the Shannon-Feinstein channel coding theorem. The reader who is seriously interested in information theory will have to go elsewhere. (McEliece’s book, [M], is a good place to start, although it omits many ideas that are of interest to the mathematical community.)

In addition to topics mentioned above, other ideas given too little mention include the Rohlin-Krieger generator theorems and the entropy theory of flows, induced maps, skew products, and group actions. No mention is made of other useful entropy ideas, such as the Slepian-Wolf coding theorems or Ziv's complexity concept, or the connection between entropy and the Kullback-Leibler distance that is so useful in hypothesis testing.

REFERENCES

- [AA] V. I. Arnold and A. Avez, *Ergodic problems of classical mechanics*, Benjamin, New York, 1968.
- [B] P. Billingsley, *Ergodic theory and information*, Wiley, New York, 1965.
- [F] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton Univ. Press, Princeton, N. J., 1981.
- [H] P. Halmos, *Lectures on ergodic theory*, Japan Mathematical Society, 1956.
- [KS] R. Kindermann and J. L. Snell, *Markov random fields and their applications*, Contemporary Math., vol. 1, Amer. Math. Soc., Providence, R. I., 1980.
- [M] R. J. McEliece, *The theory of information and coding*, Encyclopedia of Math. and Appl., vol. 3, Addison-Wesley, Reading, Mass., 1977.
- [O] D. S. Ornstein, *Ergodic theory, randomness and dynamical systems*, Yale Math. Monographs, vol. 5, Yale Univ. Press, New Haven, Conn., 1974.
- [PT] W. Parry and S. Tuncel, *Classification problems in ergodic theory*, London Math. Soc. Lecture Notes, vol. 66, Cambridge Univ. Press, Cambridge, 1982.
- [S] P. Shields, *The theory of Bernoulli shifts*, Univ. of Chicago Press, Chicago, 1973.

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