subject of interpolation itself. The translation is lucid, professionally done, and reads well. All in all, the book is a welcome addition to the literature. Wordsworth, we are sure, would have approved.

COLIN BENNETT

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Generalized solutions of Hamilton-Jacobi equations, by P. L. Lions, Research Notes in Mathematics, Vol. 69, Pitman Advanced Publishing Program, Boston, 1982, 317 pp., $24.95. ISBN 0-2730-8556-5

Introduction. The Hamilton-Jacobi equation is probably known to most engineers and physicists as a partial differential equation which pops up in the study of (Lagrangian or Hamiltonian) mechanics, yielding solutions of a system of ordinary differential equations, as its characteristics, after a variational procedure is used. It is also known, again through its relation to the calculus of variations, to people studying control theory, differential games, or other optimization problems, although it is sometimes referred to as the "Bellman equation" in these contexts.

The last thirty years has seen the rise of a new interest in the Hamilton-Jacobi equation. With the rise of computers and new numerical techniques, the failure of classical smooth solutions to describe physical situations except in limited (local) domains, and the needs of mathematical modeling, aerospace engineering, and other applications to have solutions described everywhere, many mathematicians have become interested in global solutions (whatever that means). As nearly the most general first order partial differential equation, and as an equation for which global results were possible, the Hamilton-Jacobi equation became a natural target for mathematicians studying global solutions.

In order to clarify the object of interest a little better, let us define the Hamilton-Jacobi equation. In its most familiar classical form, the Hamilton-Jacobi equation is

$$\partial u/\partial t + H(t, x, Du) = 0,$$

where H is a given function, called the Hamiltonian, x is in R^n, and Du denotes the gradient of the solution, u, with respect to x. Here t is a single variable (usually called "time"). The separation of the distinguished variable "t" from the gradient, Du, in H, makes the Hamilton-Jacobi equation much easier to handle than the general first order equation. The Cauchy (or initial value) problem is always noncharacteristic, thus amenable to solution. This same separation of t also makes the Hamilton-Jacobi equation essentially an evolution equation, thus allows a mass of evolution equation techniques to be brought to bear.

The Hamilton-Jacobi equation, as defined by Professor Lions, is

$$H(x, u, Du) = 0,$$

where "t" has been absorbed into x. That is, there is no distinguished variable, and we have the most general first order partial differential equation. Global results for this most general equation are much more recent, several of them being presented for the first time in Professor Lions' book.

Brief history. In the early 1950s, Hopf [**21**], and Cole [**9**], among others, began the study of global solutions of partial differential equations. Actually A. R. Forsyth did some such studies about fifty years sooner (see [**20**]), but his work was buried in a voluminous six-volume treatise, and was largely ignored. P. D. Lax contributed several papers during the 1950s, particularly in the area of hyperbolic systems and shocks. O. A. Oleinik obtained the first global results applicable to the Hamilton-Jacobi equation, giving both existence and uniqueness theorems, in the late 1950s (see [**28**]). The mid 1960s saw much activity in this area. Conway and Hopf [**10**] introduced the variational method in global problems, Hopf [**22**] used a very general envelope lemma and even obtained some results in a nonconvex case. Aizawa and Kikuchi obtained some results for more general boundaries (mixed problems), and Kruzkov gave results for quite general Cauchy problems in several variables (see [**23**]). Several other authors produced significant papers during the 1960s. A few of these are included in the references. Perhaps two of the most significant papers of this period were those of Fleming [**19**], and Douglis [**12**]. Douglis used a differential-difference scheme to give both existence and uniqueness results for very general Cauchy problems, and Fleming used both the variational method and "vanishing viscosity" techniques to solve quite general Cauchy problems.

In the early 1970s, Benton [**4, 5, 6**] developed general existence theorems for the convex case by pushing the variational method about as far as it would go. These results handled completely general boundaries (including the Cauchy and "mixed" problems), and clarified the role of various "compatibility" conditions between the Hamiltonian and the boundary data. However, this work included no uniqueness results, and did not consider the nonconvex case. But Feltus [**18**] did obtain some uniqueness results for boundary value problems. Elliott and Kalton [**14–17**] produced interesting results in the study of differential games. Perhaps the most interesting development of the mid 1970s was the introduction of modern functional analysis into the arena of Hamilton-Jacobi equations by treating them as evolution equations and applying the semigroup methods of Crandall and Liggett. Tulane University students, particularly Burch [**7, 8**], working under J. A. Goldstein led the way here, along with Aizawa [**1, 2**].

The last few years have seen most of the methods of partial differential equations brought to bear upon the Hamilton-Jacobi equation, including difference schemes, differential-difference schemes, variational methods, semigroup methods, layering methods, smoothing, and "vanishing viscosity". (This last is a sort of pertubation method, introducing a small second-order term which goes to zero, solving the resulting equation, perhaps by stochastic methods, and studying the limiting case.) Some of the above-mentioned authors have been active in these areas, as we have several others, including, of course, Professor Lions himself. As the list of 132 references in Professor

Lions' book is quite adequate, no further authors will be mentioned here. It might be pointed out though, that this profusion of methods in itself makes the Hamilton-Jacobi equation an interesting object of study.

This brief history has not been intended to be complete, and several interesting papers have been omitted, particularly some of the more recent ones. This reviewer apologizes to those authors who escaped mention because of the reviewer's unfamiliarity with their work. The references of Professor Lions, and of Benton [**6**], should offer a much more complete view of this area of mathematics.

Results. Professor Lions' book covers the whole range of studies of the Hamilton-Jacobi equation, although several other references will need to be studied before the novice can understand all of the work completely. The book begins by summarizing the major techniques used in this area, including optimal control theory and the vanishing viscosity method, as well as a brief look at the possibility of using semigroup methods on the Hamilton-Jacobi equation.

The major part of the book is divided into two sections. The first studies the boundary value, or Dirichlet, problem, while the second studies the initial value, or Cauchy, problem. These two sections get approximately equal coverage. However, by Cauchy problem, Professor Lions refers to "cylindrical" domains, with boundary data given both for $t = 0$, and on a fixed spatial boundary for all $t > 0$. Thus he uses the word "Cauchy" to mean the less general Hamilton-Jacobi equation with the variable "t" separated out. Therefore all of his results for the Dirichlet problem apply equally to the Cauchy problem, and the Cauchy problem can be carried even further than the Dirichlet problem.

For the Dirichlet problem, the book treats convex Hamiltonians first. For the uninitiated, we should mention that "convex", in the world of Hamilton-Jacobi equations, means convex (in the usual sense) in the variable $p = Du$. Most of the theorems are proved for a simplified Hamiltonian, and the general case given as references or by noting the necessary extensions to the proofs. This is no restriction in the generality of the results, but rather is introduced simply for ease of presentation. Quite general existence results are given, then uniqueness and stability are covered. The author then moves on to general (nonconvex) Hamiltonians, and gives general existence and uniqueness results. Various related problems, such as the existence of classical solutions, Neumann problems, regularity of solutions near the boundary, and the relations to optimal control theory, are then presented.

The treatment of the Cauchy problem is similar, handling existence, uniqueness, and stability of solutions, first for convex Hamiltonians, and then for the general case. This section also studies the propagation of singularities, which has been an area of interest since the early 1950s, as well as relations to specific hyperbolic systems of equations.

Both sections of the book include mention of questions not addressed by the book, such as numerical solutions, and various applications.

Summary. Professor Lions' book is probably the only comprehensive treatment of first order partial differential equations now available. Although first order equations are not the equations traditionally studied in courses on partial differential equations, physics, or engineering, they do have many applications, and offer a rich variety of methods. Thus anyone interested in partial differential equations should find the first order equation of interest, and Professor Lions' book a good introduction to this area. The format (basically typed, with no right justification or italics, but underlining) makes some complicated hypotheses and equations a little difficult to read. Much more mathematical background is needed than some might like. (One should know about measures, distributions, Banach spaces, Sobolov spaces, stochastic processes, strongly continuous semigroups, etc.) However, the comprehensive nature of the book, and the depth of the results, require this background. The book includes a good bibliography and a detailed table of contents, but no index.

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Surfaces and planar discontinuous groups, by Heiner Zieschang, Elmar Vogt, and Hans-Dieter Coldewey, Lecture Notes in Math., vol. 835, Springer-Verlag, Berlin, 1980, x + 334 pp., $21.00. ISBN 0-3871-0024-5

There is a basic connection between the study of combinatorial group theory and Riemann surfaces which arises as follows: If S is a compact Riemann surface of genus $g \geqslant 2$, then its universal cover is U, the unit disc in the complex plane, and S can be represented as U/Γ where Γ is the group of covering transformations. Γ is generated by $2g$ transformations $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ which satisfy the relation $\prod_{i=1}^{g}[a_i, b_i] = 1$. Here $[c,d] = cdc^{-1}d^{-1}$ and a_i and b_i are Möbius transformations which leave U invariant. The group Γ is an example of a planar discontinuous group and Γ is said to represent S. The single relation above gives a finite presentation of Γ with $2g$ generators. The situation for noncompact surfaces is more complex, but similar.

The book under review begins with combinatorial group theory and develops that part of combinatorial group theory which is relevant to Teichmüller theory and the theory of Riemann surfaces. Teichmüller theory and Riemann surfaces is an interesting field to work in because it lies at the intersection of so many fields: complex analysis, topology, algebraic geometry, differential geometry, combinatorial group theory, and geometric topology. Most of the basic theorems in the subject can be proved using the methods of any one of these fields. When one translates from the language of one field to that of another, different aspects of the theory become either more or less clear and elegant.

An example of where the combinatorial approach is especially nice is the application of the Reidemeister-Schreier methods to the theory of automorphisms of Riemann surfaces: A homeomorphism of a Riemann surface induces