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MILNOR ALGEBRAS AND EQUIVALENCE RELATIONS AMONG HOLOMORPHIC FUNCTIONS[1]

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Let $\mathcal{O}_{n+1}$ denote the ring of germs at the origin of holomorphic functions $(\mathbf{C}^{n+1},0) \to \mathbf{C}$. As a ring $\mathcal{O}_{n+1}$ has a unique maximal ideal m, the set of germs of holomorphic functions which vanish at the origin. Let G_{n+1} be the set of germs at the origin of biholomorphisms $\phi\colon (\mathbf{C}^{n+1},0) \to (\mathbf{C}^{n+1},0)$. The following are three fundamental equivalent relations in $\mathcal{O}_{n+1}$.

DEFINITION 1. Let f, g be two germs of holomorphic functions $(\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$.

(i) f is right equivalent to g if there exists a $\phi \in G_{n+1}$ such that $f = g \circ \phi$.

(ii) f is right-left equivalent to g if there exists a $\phi \in G_{n+1}$ and $\psi \in G_1$ such that $f = \psi \circ g \circ \phi$.

(iii) f is contact equivalent to g if $(V,0)$ is biholomorphic equivalent to $(W,0)$ where $V = \{z \in \mathbf{C}^{n+1}\colon f(z) = 0\}$ and $W = \{z \in \mathbf{C}^{n+1}\colon g(z) = 0\}$, i.e., there exists a $\phi \in G_{n+1}$ such that $\phi\colon (V,0) \to (W,0)$.

One of the natural and fundamental problems in complex analytic geometry is to tell when two germs of holomorphic functions $(\mathbf{C}^{n+1},0) \to (\mathbf{C},0)$ are equivalent in the sense of (i), (ii), or (iii) respectively in Definition 1. To answer the above problem, we need the following notations:

$$f^{-1}m_1 = \left\{\sum_{i\geq 1} a_i f^i\colon \sum_{i\geq 1} a_i t^i \text{ is a convergent power series in one variable}\right\},$$

$$\begin{aligned}
\Delta(f) &= \text{ideal in } \mathcal{O}_{n+1} \text{ generated by } \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n},\\
a(f) &= \{g \in m\colon \Delta(g) \subseteq \Delta(f)\},\\
\underline{R}(f) &= \{g \in m\colon g \text{ is right equivalent to } f\},\\
\underline{RL}(f) &= \{g \in m\colon g \text{ is right-left equivalent to } f\},\\
\underline{K}(f) &= \{g \in m\colon g \text{ is contact equivalent to } f\},\\
\underline{A}(f) &= \{g \in m\colon \text{the moduli algebra of } g \text{ is isomorphic to the}\\
&\qquad \text{moduli algebra of } f, \text{ i.e., } \mathcal{O}_{n+1}/(f,\Delta(f)) \cong \mathcal{O}_{n+1}/(g,\Delta(g))\},\\
\underline{B}(f) &= \{g \in m\colon \mathcal{O}_{n+1}/(f, m\Delta(f)) \cong \mathcal{O}_{n+1}/(g, m\Delta(g))\},\\
\underline{Q}(f) &= \{g \in m\colon \text{The Milnor algebra of } g \text{ is isomorphic}\\
&\qquad \text{to Milnor algebra of } f, \text{ i.e., } \mathcal{O}_{n+1}/\Delta(f) \cong \mathcal{O}_{n+1}/\Delta(g)\}.
\end{aligned}$$

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To answer the above question, it suffices to characterize the relationships between $\underline{R}(f)$, $\underline{RL}(f)$, $\underline{K}(f)$, $\underline{Q}(f)$, $\underline{A}(f)$ and $\underline{B}(f)$. It is easy to check that we have

$$\begin{array}{ccccccc} \underline{R}(f) & \subset & \underline{RL}(f) & \subset & \underline{K}(f) & \subset & \underline{A}(f) \\ & & \cap & & \cap & & \\ & & \underline{Q}(f) & & \underline{B}(f) & & \end{array}$$

THEOREM 1 (MATHER-YAU [2]). *Suppose f has isolated critical point at the origin. Then $\underline{K}(f) = \underline{A}(f) = \underline{B}(f)$.*

THEOREM 2 (SHOSHITAĬSHVILI [3]). *Suppose f has isolated critical point at the origin. Then the following statements are equivalent.*

(i) *f is right equivalent to a weighted homogeneous polynomial.*
(ii) $\underline{Q}(f) = \underline{R}(f)$.
(iii) $\overline{a(f)} \subseteq m\Delta(f)$.

THEOREM 3 (SHOSHITAĬSHVILI [3]). *Suppose f has isolated critical point at the origin. Then $\underline{Q}(f) = \underline{RL}(f)$ if and only if $f^{-1}m_1 + m\Delta(f) = a(f) + m\Delta(f)$.*

In this note we announce that we can completely characterize the relationships between $\underline{R}(f)$, $\underline{RL}(f)$, $\underline{K}(f)$, and $\underline{Q}(f)$. Theorem 9 and Corollary 11 are very interesting. They say that we can put two different "embedding differentiable structures" on a singularity.

PROPOSITION 4. *Suppose f has isolated critical point at the origin. Then the following statements are equivalent.*

(i) *f is right equivalent to a weighted homogeneous polynomial.*
(ii) $\underline{K}(f) = \underline{R}(f)$.
(iii) $f \in m\Delta(f)$.

PROPOSITION 5. *Suppose f has isolated critical point at the origin. Then the following statements are equivalent.*

(i) *f is right equivalent to a weighted homogeneous polynomial.*
(ii) $\underline{R}(f) = \underline{RL}(f)$.
(iii) $f^{-1}m_1 \subseteq m\Delta(f)$.

Propositions 4 and 5 are easy consequences of Theorem 2. We can prove that Theorem 2 remains true for arbitrary f, i.e., f may have an arbitrary nonisolated singularity at origin.

THEOREM 6. *In the following statements,* (i)$\Rightarrow$(ii) *and* (ii)$\Rightarrow$(iii).

(i) *f is right equivalent to a weighted homogeneous polynomial.*
(ii) $\underline{Q}(f) = \underline{R}(f)$.
(iii) $\overline{a(f)} \subseteq m\Delta(f)$.

As a consequence of Theorem 6, we have analogous statements as Propositions 5 and 6 for arbitrary f. The proof of Theorem 6 is based on the following Lemma.

LEMMA. *Let f, $g \in \mathcal{O}_{n+1}$. Suppose f is weighted homogeneous, i.e., there exists a holomorphic map*

$$\sigma\colon \mathbf{C}^* \times \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}, \qquad \sigma(t, z_0, \ldots, z_n) = (t^{a_0} z_0, \ldots, t^{a_n} z_n)$$

and $(f \circ \sigma)(t, z_0, \ldots, z_n) = t^d f(z_0, \ldots, z_n)$ where $a_0, \ldots, a_n$ and d are positive integers. If $\Delta(g) = \Delta(f)$, then $g \in m\Delta(g)$.

THEOREM 7. *Suppose f has isolated critical point at the origin. Then the following statements are equivalent.*

(i) $m(f) \subseteq m\Delta(f)$.

(ii) $\underline{RL}(f) = \underline{K}(f)$.

(iii) $f^{-1}m_1 + m\Delta(f) = (f) + m\Delta(f)$.

Unlike Propositions 4 and 5, Theorem 7 is new in a nontrivial sense.

DEFINITION 2. Let $f \in \mathcal{O}_{n+1}$. f is called quasi-homogeneous if $f \in m\Delta(f)$. f is called almost quasi-homogeneous if $m(f) \subseteq m\Delta(f)$.

Suppose f has isolated critical point at the origin. Then f is quasi-homogeneous if and only if f is right equivalent to a weighted homogeneous polynomial. Therefore the singularities defined by quasi-homogeneous functions have very special properties. Because of Theorem 7, we expect that isolated singularities defined by almost quasi-homogeneous functions also have special properties. The following is an example of an almost quasi-homogeneous function but not a quasi-homogeneous one.

EXAMPLE. Let $f \in \mathcal{O}_2$. $f(x, y) = x^5 + y^5 + x^3y^3$. We have

$$\underline{K}(f) = \underline{Q}(f) = \underline{RL}(f) \supsetneq \underline{R}(f).$$

f is almost quasi-homogeneous but not quasi-homogeneous.

It remains to characterize the relationship between $\underline{Q}(f)$ and $\underline{K}(f)$. The natural questions are whether $\underline{K}(f) \subset \underline{Q}(f)$ or whether $\underline{Q}(f) \subset \underline{K}(f)$? The first question is equivalent to asking whether a Milnor algebra is an invariant of a singularity, i.e., whether a Milnor algebra is, up to isomorphism, independent of the defining equation of the singularity. The second question is more important. It asks whether complex structures of isolated singularities are determined by Milnor algebras.

PROPOSITION 8. *If $\underline{K}(f) \subset \underline{Q}(f)$, then $f \in \Delta(f) + m\Delta^2(f)$ where $\Delta^2(f)$ is the ideal in $\mathcal{O}_{n+1}$ generated by all second partial derivatives of f.*

PROOF. By a tangent space argument, we have

$$(f) + m\Delta(f) \subseteq a(f) + m\Delta(f).$$

Therefore $(1 + x_0)f \in a(f) + m\Delta(f)$. Let $g \in a(f)$ and $y_j \in m$ such that

$$(1 + x_0)f = g + \sum_{j=0}^{n} y_j \frac{\partial f}{\partial x_j},$$

$$f + (1 + x_0)\frac{\partial f}{\partial x_0} = \frac{\partial g}{\partial x_0} + \sum_{j=0}^{n} \frac{\partial y_j}{\partial x_0}\frac{\partial f}{\partial x_j} + \sum_{j=0}^{n} y_j \frac{\partial^2 f}{\partial x_0 \partial x_j}.$$

Since $\partial g/\partial x_0 \in a(f)$ by definition of $a(f)$, we have $f \in \Delta(f) + m\Delta^2(f)$. Q.E.D.

The following example is due to Mather.

EXAMPLE. There exists an f such that $f(x,y) = \sum_{i=1}^{6} x^{a_i} y^{b_i}$ where

$$\max(a_i - a_j, b_i - b_j) \geq 3 \quad \text{for } i \neq j \tag{1}$$

and $f \notin \Delta(f) + \Delta^2(f)$; in particular, $\underline{K}(f) \not\subseteq \underline{Q}(f)$ by Proposition 8.

To prove this, it is easily seen that a necessary condition for $f \in \Delta(f) + \Delta^2(f)$ is that the six equations

$$1 = a_i x_1 + b_i x_2 + a_i(a_i - 1)x_3 + a_i b_i x_4 + b_i(b_i - 1)x_5,$$

$i = 1, \ldots, 6$, have solutions $x_1, \ldots, x_5 \in \mathbf{C}$. This is impossible if the matrix

$$\begin{vmatrix} 1 & a_1 & b_1 & a_1(a_1-1) & a_1b_1 & b_1(b_1-1) \\ 1 & a_2 & b_2 & a_2(a_2-1) & a_2b_2 & b_2(b_2-1) \\ 1 & a_3 & b_3 & a_3(a_3-1) & a_3b_3 & b_3(b_3-1) \\ 1 & a_4 & b_4 & a_4(a_4-1) & a_4b_4 & b_4(b_4-1) \\ 1 & a_5 & b_5 & a_5(a_5-1) & a_5b_5 & b_5(b_5-1) \\ 1 & a_6 & b_6 & a_6(a_6-1) & a_6b_6 & b_6(b_6-1) \end{vmatrix}$$

is nonsingular. Let $\Delta(\tilde{a}, \tilde{b})$ denote its determinant.

Thus, it is enough to show that there exist $\tilde{a} = (a_1, \ldots, a_6) \in \mathbf{Z}_+^6$ and $\tilde{b} = (b_1, \ldots, b_6) \in \mathbf{Z}_+^6$ such that (1) holds and the above matrix is nonsingular. (Notation: $\mathbf{Z}_+ = \{\text{nonnegative integers}\}$, $\mathbf{Q}_+ = \{\text{nonnegative rationals}\}$.) Since the functions 1, a, b, $a(a-1)$, ab, $b(b-1)$ are linearly independent, there exist $\tilde{a} = (a_1, \ldots, a_6) \in \mathbf{Q}_+^6$ and $\tilde{b} = (b_1, \ldots, b_6) \in \mathbf{Q}_+^6$ such that (1) holds and $\Delta(\tilde{a}, \tilde{b}) \neq 0$. Then $\Delta(\lambda\tilde{a}, \lambda\tilde{b})$ is a polynomial in λ and is not identically zero. Hence there exists $\lambda \geq 1$, for which $\lambda\tilde{a} \in \mathbf{Z}_+^6$, $\lambda\tilde{b} \in \mathbf{Z}_+^6$, and $\Delta(\lambda\tilde{a}, \lambda\tilde{b}) \neq 0$.

THEOREM 9. *For any $n > 1$, let $f(x_1, \ldots, x_n)$ be a non-quasi-homogeneous function with isolated critical point at the origin. Let $F(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n)$. Then there exists $G(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{C}\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ such that $\Delta(F) = \Delta(G)$ and $V = \{(x,y) \colon F(x,y) = 0\}$ is not C^∞ diffeomorphic equivalent to $W = \{(x,y) \colon G(x,y) = 0\}$ although V is homeomorphic equivalent to W, i.e., there does not exist a diffeomorphism $H \colon (\mathbf{C}^{2n}, 0) \to (\mathbf{C}^{2n}, 0)$ such that $H \colon (V, 0) \to (W, 0)$.*

COROLLARY 10. $\underline{Q}(F) \not\subseteq \underline{K}(f)$.

COROLLARY 11. *For any $n > 1$, there exists a one-parameter family of non-quasi-homogeneous isolated singularities such that the Milnor algebras corresponding to the different parameters are the same; however, the diffeomorphic types of the singularities are not the same.*

We should remark that in [**4**] we have given criterions for two germs of holomorphic functions with isolated critical points to be right-left equivalent or right equivalent.

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