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FP_∞ GROUPS AND HNN EXTENSIONS[1]

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A group G is said to be of type FP_∞ if the $\mathbf{Z}G$-module $\mathbf{Z}$ admits a projective resolution (P_i) of finite type (i.e., with each P_i finitely generated). If G is finitely presented, this is equivalent by Wall [**5**, **6**] to the existence of an Eilenberg-Mac Lane complex $K(G,1)$ of finite type (i.e., with finitely many cells in every dimension). Up to now, all known torsion-free groups of type FP_∞ have had finite cohomological dimension; in fact, they have admitted a *finite* $K(G,1)$-complex. We announce here the first known example of a torsion-free group of type FP_∞ with infinite cohomological dimension. This solves Wall's problem F11 [**7**]. We show in addition that our group, which we denote by F, satisfies $H^n(F,\mathbf{Z}F)=0$ for all n. As far as we know, F is also the first example of an FP_∞ group with this property. The vanishing of $H^*(F,\mathbf{Z}F)$ is a consequence of results of independent interest concerning the cohomology of HNN extensions (or, more generally, fundamental groups of graphs of groups) with free coefficients.

The group F is defined by the presentation $\langle x_0,x_1,x_2,\ldots; x_i^{-1}x_nx_i=x_{n+1}$ for $i<n\rangle$. It has previously arisen in two contexts: (i) finitely presented infinite simple groups (R. J. Thompson [unpublished]); and (ii) unsplittable free-homotopy idempotents (Freyd and Heller [**3**], Dydak and Minc [**2**]). F was previously known to be finitely presented, torsion-free, and of infinite cohomological dimension. (In fact, F has a subgroup which is free abelian of infinite rank.) Our contribution, therefore, is

THEOREM 1. *The group F described above is of type FP_∞ and satisfies* $H^*(F,\mathbf{Z}F)=0$.

The proof that F is of type FP_∞ goes as follows. Let $\phi\colon F\to F$ be the shift map, $\phi(x_i)=x_{i+1}$. Note that $\phi^2=T_{x_0}\circ\phi$, where T_{x_0} is the conjugation map $x\mapsto x_0^{-1}xx_0$; thus ϕ is idempotent up to conjugacy. We construct the universal example of a semicubical complex K with (a) a free right F-action; (b) a basepoint-preserving cubical endomorphism $\psi\colon K\to K$ compatible with ϕ; and (c) a homotopy from ψ^2 to $\rho_{x_0}\circ\psi$ compatible with ϕ^2, where $\rho_{x_0}(e)=ex_0$. (The motivation for this comes from (ii) above; K should be thought of as the universal cover of a complex with a free-homotopy idempotent, and ψ should be thought of as a lift of the idempotent to K.) We prove by a direct combinatorial argument that K is acyclic; the chain complex C of K is therefore a free resolution of $\mathbf{Z}$ over $\mathbf{Z}F$. Unfortunately, C is not of finite type. But we are able to find a contractible chain subcomplex $D\subset C$ such

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that the quotient $P = C/D$ is free of finite rank in every dimension. This is then the desired finite type resolution.

One can give a direct description of this resolution P. It is free of rank 1 in dimension 0 and free of rank 2 in every positive dimension. Moreover, there are formulas for computing the boundary operator inductively. In spite of this explicit description of P, however, we know of no proof of its acyclicity other than the one outlined above which uses the "big" resolution C.

We turn now to the assertion in Theorem 1 that $H^*(F, \mathbf{Z}F) = 0$. Let F_1 be the subgroup of F generated by the x_i for $i \geq 1$. It is known that F_1 is isomorphic to F via the shift map ϕ and that F is the HNN extension of F_1 with respect to the monomorphism $\phi \mid F_1$ (with x_0 as the stable letter). We now appeal to a general result about HNN extensions in which the base group and associated subgroups satisfy appropriate finiteness conditions; for simplicity, we will state a special case of this result which suffices for the present application.

THEOREM 2. *Let G be an HNN extension in which the base group G_1 and associated subgroups A and B are of type FP_∞. Assume that one of the associated subgroups, say A, has the property that the restriction map $H^*(G_1, \mathbf{Z}G_1) \to H^*(A, \mathbf{Z}G_1)$ is a monomorphism. Then in the Mayer-Vietoris sequence*

$$\cdots \to H^q(G, \mathbf{Z}G) \to H^q(G_1, \mathbf{Z}G) \xrightarrow{\alpha} H^q(A, \mathbf{Z}G) \to \cdots$$

the map α is a monomorphism.

This generalizes a result of Bieri [**1**, Theorem 6.6], in which A and B were both assumed to be of finite index in G_1. Note that our hypothesis about the restriction map holds whenever *one* of these subgroups is of finite index. In particular, it holds when $G_1 = A$, which is the case in our present application with $G = F$ and $G_1 = A = F_1$. If we now assume inductively that $H^{q-1}(F, \mathbf{Z}F) = 0$, it follows that $H^{q-1}(F, L) = 0$ for any free $\mathbf{Z}F$-module L. Since $F_1 \approx F$, this yields $H^{q-1}(F_1, \mathbf{Z}F) = 0$, so the Mayer-Vietoris sequence takes the form

$$0 \to H^q(F, \mathbf{Z}F) \to H^q(F_1, \mathbf{Z}F) \xrightarrow{\alpha} H^q(F_1, \mathbf{Z}F) \to \cdots.$$

Theorem 2 now implies that $H^q(F, \mathbf{Z}F) = 0$, as required. This completes the sketch of the proof of Theorem 1.

To prove Theorem 2, one can give a normal form argument. Alternatively, there is a proof which makes use of the tree associated to the HNN extension [**4**]. This second proof is of interest because it leads to a generalization of Theorem 2 to fundamental groups of graphs of groups, as follows.

Let G be the fundamental group of a finite graph of groups [**4**]. We will assume for simplicity that the vertex and edge groups are all of type FP_∞, although this hypothesis can be weakened. Let X be the associated tree. For each integer q there is a "coefficient system" $\mathcal{D}^q$ on X which associates to each vertex or edge σ of X the group $H^q(G_\sigma, \mathbf{Z}G_\sigma)$, where G_σ is the isotropy subgroup of G at σ, and which associates to each incidence relation "v is a vertex of e" the map $H^q(G_v, \mathbf{Z}G_v) \to H^q(G_e, \mathbf{Z}G_e)$ induced by the inclusion $G_e \hookrightarrow G_v$ and the canonical projection $\mathbf{Z}G_v \to \mathbf{Z}G_e$. Our hypotheses imply that this coefficient system is locally finite in a suitable sense, so that we can form the complex $C_c^*(X, \mathcal{D}^q)$ of cochains with compact supports and hence the cohomology groups $H_c^*(X, \mathcal{D}^q)$. One now verifies that in the Mayer-Vietoris sequence with $\mathbf{Z}G$-coefficients, the map analogous to the map α of Theorem 2 can be identified with the coboundary map $C_c^0(X, \mathcal{D}^q) \to C_c^1(X, \mathcal{D}^q)$. The content of Theorem 2, then, is that (under the hypotheses of the latter) $H_c^0(X, \mathcal{D}^q) = 0$. It is not hard to verify this by directly checking definitions. In the general case this discussion yields

THEOREM 3. *Let G be the fundamental group of a finite graph of groups of type FP_∞ as above. Then there is a short exact sequence*

$$0 \to H_c^1(X, \mathcal{D}^{q-1}) \to H^q(G, \mathbf{Z}G) \to H_c^0(X, \mathcal{D}^q) \to 0.$$

In particular, this shows that $H^q(G, \mathbf{Z}G) \approx H_c^1(X, \mathcal{D}^{q-1})$ under the hypotheses of Theorem 2. We will give further generalizations and applications of Theorem 3 elsewhere.

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