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Theory of group representations, by M. A. Naĭmark and A. I. Štern, translated by Elizabeth Hewitt, translation edited by Edwin Hewitt, Grundlehren der mathematischen Wissenschaften, vol. 246, Springer-Verlag, Berlin, Heidelberg, and New York, 1982, ix + 568 pp., $59.00. ISBN 0-3879-0602-9

A group representation is a homomorphism $G \to \mathrm{GL}(V)$, from a given group G into the group of invertible linear operators on some vector space V (usually, but not always, over $\mathbf{C}$). The modern theory of such representations first came into being in a remarkable series of papers by Frobenius in 1896–1900 (which, incidentally, still make excellent reading today; see [**4**]). Frobenius and his immediate followers (notably Schur and Burnside) dealt with finite groups, but their ideas were soon carried over to compact groups, where they blossomed in the 1920s into the beautiful Cartan-Weyl theory of representations of compact Lie groups.

Since then, group representations have cropped up in virtually every major area of mathematics, not to mention large chunks of theoretical chemistry and physics. Thus representation theory (especially of finite groups and of Lie groups) has become not only a specialty in its own right, but also a tool that almost every mathematician or physicist can make use of. It is not surprising, therefore, that one sees more and more basic textbooks on group representations these days, written from all sorts of perspectives and for all sorts of audiences.

The late Professor M. A. Naĭmark was one of the most important pioneers in several areas of functional analysis. He is probably best remembered for his work with I. M. Gel′fand in the 1940s on the foundations of C^*-algebra theory and on the unitary representations of the classical semisimple Lie groups. As explained in the translators' preface, this monograph on representation theory was Naĭmark's last major project before his death in 1978. He enlisted as a collaborator in this effort one of his former students, A. I. Štern, who has worked mostly on unitary representations of locally compact groups.

This book is "written for advanced students, for predoctoral graduate students, and for professional scientists—mathematicians, physicists, and chemists—who desire to study the foundations of the theory of finite-dimensional representations of groups". A broad audience indeed! No wonder, then, that this book is almost 600 pages long and treats many different topics (some sketchily, some in great detail) at many different levels.

Roughly speaking, the Naĭmark-Štern book is three books in one. Chapters I and II deal with representations of finite groups. Chapters III–VII, which also are the most original part of the book (where Naĭmark's own approach to the classical groups is most apparent), deal with foundational aspects of topological groups, the Peter-Weyl theory of harmonic analysis on compact groups, Lie's Theorem, and finite-dimensional representations of the classical groups. Chapters VIII–XII, which were largely written by Štern, are basically a textbook in Lie groups, Lie algebras, and their finite-dimensional representations.

This book will not please all of the audiences for which it is supposedly intended, but it does have several good features. Clearly there is no other book covering all of the topics just listed, and the working mathematician might find it valuable as a reference work, especially if one (as this reviewer has tried to do a few times) ever teaches an elementary graduate course in representation theory. Aside from this, a mathematician who already knows representation theory fairly well might still find it instructive to read the treatment of Young tableaux (here called "Young schemes") in §II.3, the development of the representation theory of $SL(2, \mathbf{F}_q)$ in §II.5, and the very nice (and quite elementary) exposition of the finite-dimensional representation theory of $GL(n, \mathbf{C})$ and the other classical complex matrix groups in §V.3 and Chapters VI and VII. These sections are really the high points of the book, in that they cover important classical topics (which in more sophisticated books would have to be dug out as special cases of more general theories) from an elementary yet elegant point of view.

On the other hand, mathematicians who already know some representation theory are not really the constituency for which this book was written. After all, this is an elementary textbook; it begins on page 1 with the definition of a group! The authors have been careful not to assume much background on the part of the reader; the only real prerequisite is a knowledge of linear algebra over $\mathbf{R}$ and $\mathbf{C}$ up through Jordan canonical form. General topology, as well as the Stone-Weierstrass Theorem and the theory of covering spaces, are developed from scratch. Somewhat surprisingly (granted the level of the rest of the book), Chapter IV requires knowledge of existence of Haar measure (which on a compact group is quite easy) and, at one point, the theory of Hilbert-Schmidt integral operators. But on the whole, the book could be read by most first- or second-year mathematics graduate students and by most physicists and chemists.

It seems appropriate, then, to compare Naĭmark-Štern with the competing textbooks. One comparison that readily comes to mind is with Kirillov's *Elements of the Theory of Representations* **[10]**, since both texts are products of the "Russian school" and were written at about the same time. In fact, Naĭmark thanks Kirillov in his preface for suggesting various improvements to the manuscript. Nevertheless, the two texts have very little in common. The Naĭmark-Štern book is more elementary and methodical, and sticks largely to the subject of finite-dimensional representations. It is designed to teach classical mathematics as related to representations theory. Kirillov's text, on the other hand, is not really designed to teach anything systematically, but tries instead to give the reader some feel for some of the methods, ideas, and goals of modern representation theory (especially, the infinite-dimensional unitary representations of Lie groups). A first- or second-year graduate student would surely be baffled by much of Kirillov's book since it rarely proves anything and often sloughs over serious technical problems; yet he or she would be bound to sense some of the author's enthusiasm for the subject, the feeling that this is a subject which is very much alive. Naĭmark-Štern, though more carefully written, is drier and, frankly, duller. The student who reads this book would have either to have a good teacher or else to be totally self-motivated.

To the student (in either math, physics, or chemistry) who wants to learn the basics of the representation theory of finite groups (over the complexes—modular representations are another game of much greater difficulty, at least at first), I would suggest beginning with Parts I and II of Serre's little book [**14**]. After that, Chapters I and II of Naĭmark-Štern might be pleasant reading. However, starting with Naĭmark-Štern would probably be a mistake. For one thing, the treatment is almost totally unmotivated; nowhere does one find out *why* one should study representations of a finite group. For instance, there is no mention of the applications (or even of the applicability) of group representations in crystallography, quantum chemistry, or physics, nor of applications internal to mathematics (in geometry/topology of transformation groups, in algebraic geometry and number theory, or in the structure theory of finite groups). Secondly, one learns nothing of the real techniques of the subject: restriction to normal subgroups and the "little group" method, double coset formulae for intertwining numbers, computational tricks for computing characters, etc. Thirdly, the only nontrivial examples for which anything is computed are the symmetric groups and $SL(2, \mathbf{F}_q)$ at the end of Chapter II. Simpler examples like dihedral and quaternion groups are left to the reader, and when asked to find the characters of, say, A_4 on p. 54, the reader has not yet been taught any method for computing anything. Finally, the hardest basic result in the theory, the fact that the dimension of each irreducible representation divides the order of the group, is left to the exercises on p. 80 (with no hint given). Any student who could do this problem unaided probably wouldn't need the book altogether. Curiously, this problem is immediately followed by the easy exercise of deducing from this theorem the fact that any group of order p^2 is commutative (of course this is not the most direct proof). But more sophisticated applications (like the "$p^a q^b$" theorem) aren't discussed. It's also too bad that other simple applications of induced representations aren't discussed—for instance, Blichfeldt's theorem that all irreducible representations of a finite nilpotent group are monomial, and the partial converse that a finite group all of whose irreducible representations are monomial is solvable [**10**, §13.3].

As the reader will certainly have gathered by now, the strength of this book lies in its treatment of the classical groups, not in the treatment of finite groups. The discussion in Chapter IV of harmonic analysis on compact groups is quite readable and pleasant, but this is by now such standard material that many other good references are available (for instance, [**2** and **6**]). Chapters V–VII, however, contain much useful information that is hard to find in other books at this level. For instance, the formulation and the proof of Lie's Theorem are not the usual ones using Lie algebras, but are very close to those usually given for the Lie-Kolchin Theorem in algebraic group theory, and the highest-weight classification of the irreducible representations of $U(n)$ is done without Lie algebras, essentially by using the main idea in the proof of the Borel-Weil Theorem without saying so. No geometry or topology is used, however, and the only technique required is the "Gauss decomposition" of a "regular" matrix in $\mathrm{GL}(n, \mathbf{C})$ as a product of a lower-triangular and an upper-triangular matrix. (This is a special case of "Bruhat decomposition," phrased in terms of elementary linear algebra.) The only real problem with this part of the book is that, once again, motivation and applications are lacking. The important fact that representations of $U(n)$ and holomorphic representations of $\mathrm{GL}(n, \mathbf{C})$ are essentially the same is only mentioned at the end of Chapter VI (one direction is nicely done in Theorem 1 on p. 275, but the other half of this fact only appears in the Remark on p. 282). For this reason, it might take the uninitiated reader a long time to realize that Chapters V and VI have anything to do with Chapter IV.

As we pointed out earlier, Chapters VIII–XII of the Naĭmark-Štern book (roughly half the text) could stand as an independent textbook on Lie groups, Lie algebras, and their representations. This is again a subject on which many texts have been written. I personally prefer the book by Varadarajan [**15**], which was reviewed in detail in this *Bulletin* a few years ago by Anthony Knapp [**11**], or else (for the reader who wants only the Lie algebra theory) the book by Humphreys [**9**]. Štern's exposition is quite readable, and doesn't do too badly according to the standards which Knapp proposed for judging a book of this sort, but it does omit the proofs of a few key theorems: existence of an analytic subgroup corresponding to a Lie subalgebra of the Lie algebra of a Lie group (Knapp's "first tricky point"), conjugacy of Cartan subalgebras in a Lie algebra over $\mathbf{C}$ (which is not even mentioned), and construction of the exceptional simple algebras. I don't mean to imply, however, that many details are omitted in Chapters X and XI. On the contrary, one will find here most of the standard theory, including useful but more advanced topics such as Levi-Mal′cev and Iwasawa decomposition, automatic analyticity of Lie group homomorphisms, and the formula for the differential of the exponential map. Some of the material from Chapters V–VII (especially, Lie's Theorem and Weyl's "unitary trick") is also redone from a new point of view. The book concludes with a short chapter which merges the Lie-algebra methods of Chapters X–XI with the techniques of Chapter VI to give a realization of the irreducible finite-dimensional representations of a connected complex semisimple Lie group. This section is rather similar to parts of Želobenko's book [**17**] —see again [**11**].

As noted in the preface, this book had previously been translated into French in 1979, as part of the Soviet-sponsored "Éditions de Moscou" series. It was a tremendous bargain at such a low price. The English version by Springer is more sumptuous, but also considerably more expensive. A definite advantage of the English edition is that Edwin Hewitt (who of course is familiar from [**6**] with the problems of writing comprehensive textbooks on harmonic analysis) has added a few "editor's notes" clarifying some details, and has corrected some misprints in the earlier editions. Unfortunately, Hewitt hasn't quite gone far enough. There are still some misprints which were present in the French edition (e.g., G for $\tilde{G}$ on *l*.3 of p. 541, $h \to \lambda_{1g}(h)$ for $g \to \lambda_{1g}$ on *l*.7 and Y for Y_1 on *l*.14 of p. 244), and some new ones have crept in (e.g., χ for $\bar{\chi}$ in formula (2.6.28) on p. 199, and "unimodular orthogonal group" as a new name for $SL(n+1, \mathbf{C})$ on p. 283). In simplifying the proof of the Stone-Weierstrass Theorem on p. 173, the editor forgot to change formula (1.4.1), and so left the paragraph a bit muddled. More serious than the misprints are some unfortunate choices of terminology which come from trying to be too faithful to the original. A few examples: *Normalteiler* and *quadratintegrierbar* would have been good words for a German translation, but who uses (in English) "normal divisor" for "normal subgroup" or "quadratically integrable" for "square-integrable"? And why must the book use P_n instead of A_n for the alternating group,[1] $\mathbf{N}$ instead of $\mathbf{Z}$ for the group of integers, $\mathbf{R}^1$ instead of $\mathbf{R}$ for the real line, $\mathbf{R}_0$ for $\mathbf{R}^\times$ (the multiplicative group), "polylinear" for "multilinear" on p. 279, "differentiation" for "derivation" (of an algebra) on p. 368? It's similarly unfortunate that the authors call sesquilinear forms "bilinear" on p. 26, since "bilinear" on p. 283 is supposed to revert to its usual meaning. A few changes in language and notation, as well as the addition of a notational index, would have made the book more useful to the English-speaking student.

All in all, though, these complaints are minor, and this book could still be used successfully as a textbook, or better, as a reading supplement, for a first course in Lie groups or representation theory. Exercises would have to be added from other books. Applications of the theory are lacking, but then one always faces the difficulty that it's impossible to go into applications in any detail without assuming some knowledge of, or interest in, other fields.

I would hope that a student wouldn't come away from this book with the feeling that the representation theory of Lie groups is an elegant but dead subject. It is true that most of the theory presented here was worked out by 1930 (except for the "global" theory of Lie groups, which came a bit later), but much of it is still being applied in new situations. I shall conclude by mentioning a few directions in which the interested student might proceed concurrently with, or after finishing, this book.

An obvious sequel would be the subject of infinite-dimensional representations of Lie groups, with emphasis on unitary representations and their applications. This is, for instance, the main subject of Kirillov's book [**10**] or of Mackey's survey [**13**]. It also was one of Naĭmark's main areas of interest, and would have been the subject of a volume he had intended to write after completing this one. This field has grown so much in the last twenty years that it is now almost impossible for anyone to master all of it. Thus my personal feeling is that beginners would be best advised first to learn the basic examples of the Heisenberg group and $SL(2, \mathbf{R})$ in detail, following, say, [**7** and **12**]. Then they should learn the basic analytic machinery (essentials of Banach

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[1]Admittedly, A_n will be used later for $SL(n+1)$, but this is less likely to cause confusion than deviation from standard notations.

algebras, induced representations, etc.) and begin to concentrate on a subarea (such as nilpotent Lie groups and PDE applications, geometric quantization and the "orbit method," analysis on semisimple Lie groups and symmetric spaces, algebraic theory of the representations of semisimple groups, or p-adic and adelic groups) that matches their own interests and tastes.

There is also much interesting mathematics that depends only on the finite-dimensional representation theory developed in this book. Even leaving aside the obvious importance of representations of finite groups for algebra and arithmetic, one has the applications of representations of compact Lie groups to special functions (see, e.g., [**3**]), to differential geometry and harmonic analysis on compact symmetric spaces [**5**], to the classification of spherical and other space forms [**16**], to the topology of transformation groups [**8**], and of course to quantization of angular momentum and fundamental particle theory (see again [**17**, **11**], and the vast "Lie groups for physicists" literature). Many of the applications to geometry and topology depend only on one simple idea: if a Lie group G acts smoothly on a manifold M, then at any point p of M, the differential of the action gives a linear representation of the isotropy group G_p on the tangent space T_pM. The group G will also act linearly on spaces of functions and differential forms on M. Analysis of these various representations often says a lot about the geometry of the situation, especially when there are relatively few orbit types. One particularly elegant application is Bott's index theorem for homogeneous elliptic operators [**1**], which can serve as a nice introduction to the harder index theorems of differential geometry. All of these topics can be pushed to the research frontier; yet (at least in certain simple but illuminating examples) they are accessible to anyone who has learned the fundamental facts about compact Lie groups, as covered by Naĭmark-Štern or the other standard texts.

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Rings that are nearly associative, by K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov and A. I. Shirshov, translated by Harry F. Smith, Academic Press, New York, 1982, xi + 371 pp., $6.00. ISBN 0-1277-9850-1

The study of nonassociative algebras was originally motivated by certain problems in physics and other branches of mathematics, and even today the main motivation for studying some problems in the area is the applications. However, most types of nonassociative algebras are now studied more for their own sake.

The first class of nonassociative algebras to be investigated systematically was the class of Lie algebras, which arose out of the study of Lie groups. A nonassociative algebra L with product $[\,,]$ is defined to be a Lie algebra if it satisfies the identities

$$[x, y] = -[y, x], \qquad [[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

If A is any associative algebra, we define $A^{(-)}$ to be the algebra obtained from A by replacing the associative multiplication in A by the new multiplication $[\,,]$ defined in terms of the associative multiplication by $[a, b] = ab - ba$. Then $A^{(-)}$ is a Lie algebra, and so is any subalgebra of $A^{(-)}$ (that is, any subspace of A closed under $[\,,]$). Conversely, any Lie algebra over a field arises as a subalgebra of an algebra $A^{(-)}$ constructed from some associative algebra A. The methods which are used for studying Lie algebras come out of this special connection with associative algebras and are thus somewhat different from the methods that work best for other classes of nonassociative algebras. Also, partly because of the nature of the principal applications of Lie algebras, the