readily comprehensible to the mathematically informed reader. This problem arises in part from the narrative and notational difficulties of explaining outmoded mathematics in modern language. A related problem, one connected to the authors' historical methodology, is their practice of examining a given mechanical argument in isolation from the wider text in which it appears. These difficulties are apparent in the opening chapter where Cannon and Dostrovsky discuss Newton's analysis of the pressure wave in Propositions XLVII–XLIX of Book Two of the *Principia*. These propositions contain Newton's celebrated calculation of the speed of sound, an estimate that was for lack of an adiabatic correction 20% below the true value. The authors' discussion is marred by an inadequate description of two of the original propositions, a failing which makes their account very difficult to follow. This is especially unfortunate since their conclusion, that Newton had at this early date grasped clearly the concept of mechanical strain, is new and ultimately convincing.

The evolution of dynamics: *vibration theory from* 1687 *to* 1742 is a substantial addition to the survey of early 18th century mechanics provided three decades ago by Clifford Truesdell in his extensive introductions to the collected works of Leonhard Euler. Despite its occasional narrative weaknesses the book is destined to become a standard source. It will be of assistance to the specialist in the history of the exact sciences who wishes to contribute to our understanding of the still largely unexplored world of 18th century mathematics. In addition, the nonspecialist with some background in vibration theory will be rewarded by a close study of its contents. Cannon and Dostrovsky state in the preface that mathematics "provides a powerful tool with which to grasp modes of thought from former times". To this one might add that the converse is also true: knowledge of earlier modes of thought provided by historical investigation serves to heighten our appreciation for the mathematics of today.

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The method of iterated tangents with applications in local Riemannian geometry, by Enrico J. White, Monographs and Studies in Mathematics, vol. 13, Pitman Publishing Incorporated, Marshfield, Mass., 1982, xx + 252 pp., $39.95. ISBN 0-2730-8515-8

The objects studied in differential geometry can alternatively be defined by using or by avoiding local coordinates. There are even definitions which can be thought of as both using and avoiding coordinates. Consider, for example, a first order partial differential operator

$$D = \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i}$$

on some open set in $\mathbf{R}^n$. The same operator in different coordinates on the same open set is

$$D = \sum_{j=1}^{n} \bar{a}_j(\bar{x})\frac{\partial}{\partial \bar{x}_j}$$

where

$$\bar{a}_j(\bar{x}) = a_i(x)\frac{\partial \bar{x}_j}{\partial x_i}$$

(summation convention henceforth). Thus a tangent field in some manifold may be defined by assigning to each coordinate chart a set of functions $a_i(x)$ in such a way that the above equation is valid on overlapping coordinate charts. Of course, one could also define a tangent vector at a point without explicitly choosing local coordinates by considering equivalence classes of parametrized curves and then letting the vector vary "smoothly" as a function of the point to define the vector fields. Let us use "intrinsic" to refer to methods and definitions such as this last one which do not rely on any explicit coordinate choices.

An intermediate approach incorporates all coordinate changes into an intrinsic space associated to M and uses that space to define the object. In this sense it both uses and avoids local coordinates. In our example of a tangent vector field only the first derivatives of the coordinate changes are relevant. So let $P_1(M)$ be the bundle of 1-jets at the origin of nonsingular maps $f\colon \mathbf{R}^n \to M$. The group $G = \mathrm{GL}(n)$ acts on $P_1(M)$ by $j_1(f)g = j_1(f \circ g)$, i.e. by matrix multiplication on the right in each fibre. We also have the usual left action of G on $\mathbf{R}^n$. Thus we may define an action of G on $P_1(M) \times \mathbf{R}^n$ by $g(j_1(f), v) = (j_1(f)g^{-1}, gv)$. Let $P_1(M) \times_G \mathbf{R}^n$ denote the orbit space of this action. This is a bundle over M with fibre dimension equal to n. Let us show that a tangent vector may be considered as a point of this bundle. Take $(t_1, \ldots, t_n)$ as the standard coordinates on $\mathbf{R}^n$ and choose some local coordinates $(x_1, \ldots, x_n)$ on M. Then $j_1(f) = (f(0), \partial f^i(0)/\partial t_j)$ and so (x, F_{ij}, v_α) may be taken as local coordinates for $P_1(M) \times \mathbf{R}^n$. In the orbit given by $(x, F_{ik}(g^{-1})_{kj}, g_{\alpha\beta}v_\beta)$ there is precisely one point with coordinates $(x, \mathrm{I}, a_\alpha)$ where I is the identity matrix. So we take these as coordinates on $P_1(M) \times_G \mathbf{R}^n$. Let the tangent vector $a_\alpha(\partial/\partial x_\alpha)$ at x correspond to the point $(x, \mathrm{I}, a_\alpha(x))$ of $P_1(M) \times_G \mathbf{R}^n$. This uses so far only that $P_1(M) \times_G \mathbf{R}^n$ and the tangent bundle $T(M)$ have the same dimension. But we mean to say more: The map

$$a_\alpha \frac{\partial}{\partial x_\alpha} \to (x, \mathrm{I}, a_\alpha)$$

does not depend on the choice of coordinates for M. Or said another way, points of $P_1(M) \times_G \mathbf{R}^n$ and of $T(M)$ when written in local coordinates transform in the same way. This is easily seen from the fact that if $j_1(\bar{f}) = \mathrm{I}$ when expressed in $\bar{x}$ coordinates in the range, then $j_1(\bar{f}) = \partial x/\partial \bar{x}$ when expressed in x coordinates in the range. Thus we could have defined a tangent vector as a point of this intrinsic bundle. This would be an example of an intermediate definition.

Now that we have vectors and vector fields, let us try to define a derivative. We seek to define a new vector field $X \vdash Y$ which should act like the "derivative of Y in the direction X". So we choose some coordinates and also some functions $\Gamma_{ij}^k(x)$. Then we may set

$$\frac{\partial}{\partial x_i} \vdash \frac{\partial}{\partial x_j} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

and

$$X_i \frac{\partial}{\partial x_i} \vdash Y_j \frac{\partial}{\partial x_j} = X_i Y_j \Gamma_{ij}^k \frac{\partial}{\partial x_k} + X_i \frac{\partial Y_j}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Now this same object in different coordinates must be given by

$$\bar{\Gamma}_{ij}^k = \frac{\partial \bar{x}_k}{\partial x_i} \frac{\partial^2 x_c}{\partial \bar{x}_j \partial \bar{x}_i} + \frac{\partial \bar{x}_k}{\partial x_c} \frac{\partial x_a}{\partial \bar{x}_i} \frac{\partial x_b}{\partial \bar{x}_j} \Gamma_{ab}^c.$$

So an "affine connection" may be defined as a collection of functions $\Gamma_{ij}^k(x)$ which transform in a certain way under change of coordinates.

Again we also have an intrinsic definition: An affine connection is a map from triples (p, X, Y), where $p \in M$, $X \in TM_p$, and Y is a vector field near p, to vectors in TM_p. This map is linear with respect to X and acts as a linear derivation with respect to Y. The image of (p, X, Y) may be denoted $\nabla_X Y$.

There is also an intermediate definition, at least if we restrict ourselves to torsion-free connections. (Recall that this means $\Gamma_{ij}^k = \Gamma_{ji}^k$ in one coordinate system and hence in all or, alternatively, $\nabla_X Y - \nabla_Y X = [X, Y]$.) The transformation law for Γ_{ij}^k shows that we must work with the bundle of 2-jets at the origin of nonsingular maps of $\mathbf{R}$ into M. Call this bundle $P_2(M)$. Following [**1**, p. 147], we shall essentially represent a torsion-free affine connection as a section of a quotient of this bundle; that is, as $P_2(M) \times_G \mathbf{R}^0$. We fix coordinates on M and take

$$j_2(f) = \left(f(0), \frac{\partial f^i}{\partial t_j}(0), \frac{\partial^2 f^i}{\partial t_j \partial t_k}(0) \right).$$

Thus for coordinates on $P_2(M)$ we have (x, u_j^i, u_{jk}^i). Again let $G = \mathrm{GL}(n)$ act by composition. So if $g = (g_{ab})$ then

$$\left(x, u_j^i, u_{jk}^i\right) g = \left(x, u_a^i g_{aj}, u_{ab}^i g_{aj} g_{bk}\right).$$

Note there is a unique $g \in G$ which takes the point (x, u_j^i, u_{jk}^i) to a point of the form $(x, \mathrm{I}, z_{jk}^i)$. Here $z_{jk}^i = z_{kj}^i$. Thus (x, z_{jk}^i) serve as coordinates for the orbit bundle $P_2(M)/G$. It is easy to see that when we start with different coordinates $\bar{x} = \bar{x}(x)$ on M then the section $(x, z_{ij}^k(x))$ becomes $(\bar{x}, \bar{z}_{ij}^k(\bar{x}))$ with

$$\bar{z}_{ij}^k = -\frac{\partial \bar{x}_k}{\partial x_c} \frac{\partial^2 x_c}{\partial \bar{x}_i \partial \bar{x}_j} + \frac{\partial \bar{x}_k}{\partial x_c} \frac{\partial x_a}{\partial \bar{x}_i} \frac{\partial x_b}{\partial \bar{x}_j} z_{ab}^c.$$

Thus a torsion-free affine connection may be defined as the "negative" of a section of $P_2(M)/G$.

Finally let us consider tensors. Classically an (r, s)-tensor written with respect to some coordinate system $(x_1, \ldots, x_n)$ is given by

$$A_{i_1 \cdots i_s}^{j_1 \cdots j_r} \quad \text{where } 1 \leqslant i_p \leqslant n,\ 1 \leqslant j_q \leqslant n.$$

In another coordinate system $(\bar{x}_1, \ldots, \bar{x}_n)$ we have the same object represented by

$$\bar{A}_{i_1 \cdots i_s}^{j_1 \cdots j_r} = A_{k_1 \cdots k_s}^{h_1 \cdots h_r} C_{k_1 i_1} \cdots C_{k_s i_s} C^{h_1 j_1} \cdots C^{h_r j_r}$$

where $C_{ki} = \partial x_k / \partial \bar{x}_i$ and $C^{hj} = \partial \bar{x}_h / \partial x_j$. See (2).

Now for an invariant definition. For any vector space V let $T_s^r(M)$ be the tensor product $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ where there are r factors of V and s factors of its dual. Construct a new bundle on M by doing this for each fibre of $T(M)$. A section of this bundle is a tensor field of type (r, s).

Again there is an intermediate definition which uses the frame bundle $P_1(M)$. Our definition for $r = 1$, $s = 0$ will coincide with the intermediate definition of the tangent bundle. Consider $\mathbf{R}^n$ as a vector space and let $T_s^r(\mathbf{R}^n)$ be the tensor product, as above. Let $G = \mathrm{GL}(n)$ act on $T_s^r(\mathbf{R}^n)$ in the usual way: It acts on the left on each $V = R$ factor and its adjoint acts on the left on each V^* factor. Then

$$T_s^r(M) = P_1(M) \times_G T_s^r(\mathbf{R}^n).$$

Now of course this can be generalized. For instance let $P = P_k(M)$ be the bundle of k-jets at the origin of nonsingular maps of $\mathbf{R}^n$ to M, $G = G_k$ the group of k-jets at the origin of diffeomorphisms leaving the origin fixed, and $E = E_k$ any vector space upon which G acts. Then the associated bundle $P \times_G E$ is a generalized tensor bundle.

The book under review is based on the observation that for an appropriate choice of E we have that $P \times_G E$ is $T(T(\cdots T(M)) \cdots)$, the kth iterated tangent space of M. The author generalizes to sections of $P \times_G E$ and of related bundles the operations so useful for tensors such as contractions, duals, and Lie derivatives. There is a notational difficulty in working with these iterated tangent spaces. The author handles this by some clever conventions and his notation is probably no more complicated than is necessary. As applications he considers some standard topics such as curves in the plane, surfaces in Euclidean 3-space, geodesics, and the Riemannian curvature tensor.

Although the author cautions us in his preface that he claims no originality in discovering uses for iterated tangents it is nevertheless disappointing to see such meager results after such a careful and detailed presentation of the associated algebra and calculus. Perhaps the author should have published some of his work in a journal or as a set of lecture notes and deferred the research monograph until his methods had proved their usefulness. As it is, this book can be recommended only to those mathematicians who already have a particular interest in iterated tangent spaces and are willing to pay a reference book price for material which might be more at home elsewhere.

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Theory of group representations, by M. A. Naĭmark and A. I. Štern, translated by Elizabeth Hewitt, translation edited by Edwin Hewitt, Grundlehren der mathematischen Wissenschaften, vol. 246, Springer-Verlag, Berlin, Heidelberg, and New York, 1982, ix + 568 pp., $59.00. ISBN 0-3879-0602-9

A group representation is a homomorphism $G \to \mathrm{GL}(V)$, from a given group G into the group of invertible linear operators on some vector space V (usually, but not always, over $\mathbf{C}$). The modern theory of such representations first came into being in a remarkable series of papers by Frobenius in 1896–1900 (which, incidentally, still make excellent reading today; see [**4**]). Frobenius and his immediate followers (notably Schur and Burnside) dealt with finite groups, but their ideas were soon carried over to compact groups, where they blossomed in the 1920s into the beautiful Cartan-Weyl theory of representations of compact Lie groups.

Since then, group representations have cropped up in virtually every major area of mathematics, not to mention large chunks of theoretical chemistry and physics. Thus representation theory (especially of finite groups and of Lie groups) has become not only a specialty in its own right, but also a tool that almost every mathematician or physicist can make use of. It is not surprising, therefore, that one sees more and more basic textbooks on group representations these days, written from all sorts of perspectives and for all sorts of audiences.

The late Professor M. A. Naĭmark was one of the most important pioneers in several areas of functional analysis. He is probably best remembered for his work with I. M. Gel′fand in the 1940s on the foundations of C^*-algebra theory and on the unitary representations of the classical semisimple Lie groups. As explained in the translators' preface, this monograph on representation theory was Naĭmark's last major project before his death in 1978. He enlisted as a collaborator in this effort one of his former students, A. I. Štern, who has worked mostly on unitary representations of locally compact groups.

This book is "written for advanced students, for predoctoral graduate students, and for professional scientists—mathematicians, physicists, and chemists—who desire to study the foundations of the theory of finite-dimensional representations of groups". A broad audience indeed! No wonder, then,