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The evolution of dynamics, vibration theory from 1687 *to* 1742, by John T. Cannon and Sigalia Dostrovsky, Studies in the History of Mathematics and Physical Sciences, Vol. 6, Springer-Verlag, New York, 1981, ix + 184 pp., $38.00. ISBN 0-3879-0626-6

The publication in 1687 of Isaac Newton's monumental treatise *Principia Mathematica* has long been regarded as the event that ushered in the modern period in mathematical physics. Newton developed a set of techniques and methods based on a geometric form of the differential and integral calculus for dealing with point-mass dynamics; he further showed how the results obtained could be applied to the motion of the solar system. Other topics studied in the

Principia include the motion of bodies in resisting fluids and the propogation of disturbances through continuous media. The latter theory, though less successful than his earlier treatment of point-mass dynamics, was nonetheless an important stimulus for future research.

The success and scope of the *Principia Mathematica* heralded the arrival of mechanics as the model for the mathematical investigation of nature, as the subject which would remain "the cutting edge of science" for the next two centuries. Indeed, the very extent of Newton's achievement has had the unfortunate effect of fostering a simplistic and incorrect view in the history of physics, according to which the entire modern edifice of "Newtonian mechanics" is identified with the contents of the *Principia*. In fact, a survey of the hundred or so years which followed the appearance of this treatise reveals that is was simply a first, if exceedingly important, step in the erection of such an edifice. The period 1687–1788 was one of energetic and creative activity involving the invention, mathematical development and application of mechanical principles and techniques by a distinguished group of researchers: Brook Taylor, the brothers Jacob and Johann Bernoulli and the latter's son Daniel, Leonhard Euler, Alexis Clairaut, Jean d'Alembert and Joseph-Louis Lagrange. In particular, it was not until the early 1750's that the cornerstone of modern dynamical theory—the analytical equations which express the general principle of linear momentum ("Newton's second law")—first appeared in Euler's study of rigid body analysis. It was these researches of Euler combined with parallel work by d'Alembert and Clairaut which laid the foundations for the classical theory handed down to us today.

The evolution of dynamics: *vibration theory from* 1687 *to* 1742 by John T. Cannon and Sigalia Dostrovsky is a contribution to our understanding of the period that follows the publication of Newton's *Principia* but precedes the appearance of Euler and d'Alembert's general methods at the middle of the next century. The authors trace the development of research on a range of problems involving small vibrations: the propogation of pressure waves through continuous media; the vibrations of strings, rings and rods; the oscillations of linked pendulums and hanging chains; the bobbing and rocking of a floating body. Their primary concern is to exhibit through detailed explication of the original sources the progress made in the analysis of such systems. A leading idea motivating the discussion is that Newton's second law during this period was successfully applied only to systems involving one degree of freedom (central force motion was an important exception). Such a unitary system arises in the problem of the center of oscillation of a rigid pendulum, in which one must determine the length of a simple pendulum that oscillates in unison with the given rigid pendulum. In more complicated systems—the linked pendulum, the vibrating string—involving many or infinitely many degrees of freedom, the mechanicians of the period had recourse to special methods. These methods either used the second law as a consistency condition on the actual motion or reduced the whole analysis to the case of a unitary system.

The authors' main point is illustrated in their account of the research of Brook Taylor and Johann Bernoulli on the vibrating string. Interest in this problem during the period stemmed in large part from a fascination with musical phenomena (the vibrating string itself was sometimes referred to as the "musical" string). Taylor's analysis appeared in his Latin treatise *Methodus Incrementorum* (1715), the same work incidentally which first presented to the world the series expansion that bears his name. Taylor employed the fluxional notation invented by Newton and favored by the British school. His analysis was recast in 1728 by Bernoulli using the familiar Leibnizian notation by then widely in use on the continent. We shall describe what is in substance Bernoulli's treatment. Assume a string is stretched along the z-axis in the $z - y$ plane from $z = 0$ to $z = l$ with tension P and mass density ρ. The string is given a small displacement from its equilibrium position; the problem is to determine the shape it assumes in the ensuing small vibrations and to calculate the period and frequency of these vibrations. Bernoulli first shows using elementary geometry that the force acting on the mass element ρdz equals $P(d^2y/dz^2)\,dz$ and acts in a direction perpendicular to the z-axis. At this point he could, if he had at his disposal the momentum principle furnished by Newton's second law, simply equate this force to the quantity $(\rho dz)(d^2y/dt^2)$ and thereby obtain the wave equation. Bernoulli, however, following Taylor before him, proceeds otherwise. He imagines that each mass element of the string vibrates as would a simple pendulum of length L (where L is to be determined). If the displacement of an element ρdz suspended from such a simple pendulum equals y then the restoring force is given by $-(g/L)(\rho dz)y$, where g is the acceleration due to gravity. This force in turn must equal the actual force calculated above:

$$(1) \qquad P(d^2y/dz^2)\,dz = -\,(g/L)(\rho dz)y,$$

from which we obtain the equation

$$(2) \qquad (d^2y/dz^2) = (-g/L)(\rho/P)y.$$

(2) may now be integrated to yield

$$(3) \qquad y = \pm c \sin\left(\sqrt{(g/L)(\rho/P)}\,z\right),$$

where we have expressed in functional notation what Bernoulli describes in geometrical language. (Bernoulli states that the solution to (2) is the curve known as the "companion to the trochoid [i.e. cycloid]", a characterization of the sine function which survived well into the 18th century and is explained in Morris Kline's *Mathematical Thought from Ancient to Modern Times* (Oxford, 1972, p. 351)). Using the end condition $y = 0$ when $z = l$ we obtain a value for L:

$$(4) \qquad L = (g\rho/P)(l^2/\pi^2).$$

Bernoulli has discovered that the vibrating string assumes the shape of a sine curve. It remains to find the period and frequency of the vibrations. This however is now straightforward since the device used in analyzing the string reduced the problem of finding these values to the analysis of the small vibrations of a simple pendulum of length L. The latter system, involving one degree of freedom, was known from Newton's second law to be governed by the equation

$$\ddot{y} = -(g/L)y.$$

The solution of this equation, $y = h\sin((\sqrt{g/L})t + \delta)$, immediately yields the desired values for the period T and frequency ν:

$$T = 2\pi\sqrt{(L/g)}, \qquad \nu = \sqrt{(g/L)}/2\pi. \tag{5}$$

Using the value for L given by (4) we obtain the final result for the vibrating string:

$$T = 2l\sqrt{(\rho/P)}, \qquad \nu = \sqrt{(P/\rho)}/2l. \tag{6}$$

Bernoulli has therefore derived by mathematical analysis the result known in musical theory of the 17th century as "Mersenne's law", asserting the proportionality of the pitch or frequency to the quantity $(\sqrt{P/\rho})/l$.

Central to Bernoulli's solution is the assumption that the elements of the string undergo small vibrations as simple pendulums all of the same period. Dostrovsky and Cannon refer to this assumption as the "pendulum condition"; it is the idea that underlies all investigations of oscillatory phenomena during this period. Use of the pendulum condition tended to be combined with certain restrictions on the motion. Thus both Taylor and Bernoulli assume in their analyses that the elements of the string arrive simultaneously from one side at the equilibrium configuation along the z-axis. As a result, they only determine the first fundamental mode. The authors suggest that the strong geometric viewpoint inherent in the geometric integration of equation (2) discouraged the investigation of higher modes and acted as an obstacle to the discovery of the principle of superposition. (The latter was eventually recognized as a fundamental law by Daniel Bernoulli during the 1750s in the famous debate over the general solution to the wave equation.)

Johann Bernoulli's analyis in 1728 of the vibrating string was only one example of a much broader interest in vibration phenomena. A problem which received increasing study toward the end of the period under consideration was the analysis of the linked pendulum and hanging chain. Key memoirs on this topic were presented to the St. Petersburg Academy during the 1730s by Johann's son Daniel and by Leonhard Euler. The solution to this problem, which required consideration of Laguerre polynomials and Bessel functions, served to focus attention on higher modes and to shift the emphasis of the investigation from geometric to analytic methods. One of the great merits of the Cannon and Dostrovsky book is that they provide, as an appendix, facsimiles of Daniel Bernoulli's two Latin memoirs of 1733–34 along with an English translation. A study of these memoirs in conjunction with the explication provided in the text should assist the reader in attaining what the authors term "a feel for physics in the age of Newton and the Bernoullis".

Cannon and Dostrovsky have mastered a large amount of difficult source material. Indeed, it is only the condensed nature of their presentation that prevents the book from being much longer. Unfortunately, the authors are not always completely successful in marshalling their material in a way that is readily comprehensible to the mathematically informed reader. This problem arises in part from the narrative and notational difficulties of explaining outmoded mathematics in modern language. A related problem, one connected to the authors' historical methodology, is their practice of examining a given mechanical argument in isolation from the wider text in which it appears. These difficulties are apparent in the opening chapter where Cannon and Dostrovsky discuss Newton's analysis of the pressure wave in Propositions XLVII–XLIX of Book Two of the *Principia*. These propositions contain Newton's celebrated calculation of the speed of sound, an estimate that was for lack of an adiabatic correction 20% below the true value. The authors' discussion is marred by an inadequate description of two of the original propositions, a failing which makes their account very difficult to follow. This is especially unfortunate since their conclusion, that Newton had at this early date grasped clearly the concept of mechanical strain, is new and ultimately convincing.

The evolution of dynamics: vibration theory from 1687 *to* 1742 is a substantial addition to the survey of early 18th century mechanics provided three decades ago by Clifford Truesdell in his extensive introductions to the collected works of Leonhard Euler. Despite its occasional narrative weaknesses the book is destined to become a standard source. It will be of assistance to the specialist in the history of the exact sciences who wishes to contribute to our understanding of the still largely unexplored world of 18th century mathematics. In addition, the nonspecialist with some background in vibration theory will be rewarded by a close study of its contents. Cannon and Dostrovsky state in the preface that mathematics "provides a powerful tool with which to grasp modes of thought from former times". To this one might add that the converse is also true: knowledge of earlier modes of thought provided by historical investigation serves to heighten our appreciation for the mathematics of today.

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The method of iterated tangents with applications in local Riemannian geometry, by Enrico J. White, Monographs and Studies in Mathematics, vol. 13, Pitman Publishing Incorporated, Marshfield, Mass., 1982, xx + 252 pp., $39.95. ISBN 0-2730-8515-8

The objects studied in differential geometry can alternatively be defined by using or by avoiding local coordinates. There are even definitions which can be thought of as both using and avoiding coordinates. Consider, for example, a first order partial differential operator

$$D = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$$