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Basic concepts of enriched category theory, by G. M. Kelly, London Mathematical Society Lecture Note Series, No. 64, Cambridge Univ. Press, New York, 1982, 245 pp., $24.95. ISBN 0-5212-8702-2

What has been going on in category theory for the last 15 years? Originally category theory appeared to be an outgrowth of homological algebra which itself developed as an aspect of algebraic topology. The historical accident of its birth has little to do, however, with the current perception of category theory as an alternative to set theory in the foundations and formulations of mathematics. What originally distinguished category theory from homological algebra was its retreat from groups to sets; i.e., its elimination of the requirement that maps between the same two objects could always be added. This ruled out exact sequences and hence derived functors so that for a while category theory not only bore little relation to homological algebra, but also had little to talk about. Fortunately adjoint functors were discovered (remembered? recognized?) and they became the main topic of study in the 1960's, which mostly centered around the notions of triples (monads), algebras for a triple, algebraic categories and equational categories. The names of Barr, Beck, Lawvere, and Linton are associated with this development, which had its culmination in the books by Gabriel-Ulmer [6] and Manes [21]. (See these for references to the original papers.) However, during the 60's there were other developments that could be characterized as "moving away from the safe shores of set theory". (Thanks to F. E. J. Linton for this image.) This movement took place in three interrelated ways.

1. Closed categories. Being able to add maps between the same two objects is clearly an important property of a category. This can be expressed in a more abstract form by requiring that for any two objects A and B in the category **C**, the set $\mathbf{C}(A, B)$ of maps from A to B in **C** should carry the structure of an

abelian group. More generally, $C(A, B)$ could have various kinds of enriched structure; i.e., could be an object in a suitable category V for all A and B. Being suitable means that V has *internal* homs; i.e., $V(X, Y)$ is a (function space) object in V for all X and Y in V. In good cases this internal hom is related to a tensor product $\otimes$ by the equation

$$V(X \otimes Y, Z) \simeq V(X, V(Y, Z)).$$

If $\otimes$ is also symmetric, then one has a symmetric monoidal closed category, which nowadays is just called a closed category. If $\otimes$ is the cartesian product then it is called cartesian closed. The categories of abelian groups, R-modules, Banach spaces, differential graded modules, etc., are closed. The categories of sets, categories, k-spaces, continuous lattices, etc., are cartesian closed. The first paper on this development—actually 17 years ago—was Linton [18] 1965, followed quickly by Eilenberg and Kelly [5] 1966 where many variants on the situation of a closed category V and a category C enriched in V are treated. It was apparent that in most mathematical uses of category theory the relevant categories had at least some features of this additional structure and so it was necessary to redevelop all of category theory in the context in which the important property of $C(A, B)$ was not that it had an underlying set (which it did) but that it was an object in another category. We shall return to this topic since it is the subject of the book under review.

2. Categories-an-sich. In [16] 1964, Lawvere gave an axiomatic description of the category of sets. This was followed in [17] 1966 by a similar description of the category of (small) cateogries. This together with the work of Ehresmann [4] suggested the possibility of starting from scratch by treating categories enriched (in the sense of closed categories) in the cartesian closed category of categories, thus bypassing sets altogether. Such a structure was envisioned as providing a foundation for mathematics, its objects including the usual categories of mathematics and its morphisms including all functorial relationships between such. Methods were then needed to recognize by these functorial relations, from the outside as it were, when a given object was in fact the category of sets, or of groups, or topological spaces, etc. Also, given a suitable "category of sets" object, one should be able to construct the appropriate categories of algebraic or topological entities. Thus two concerns seemed important:

(i) Determine the structure of suitable categories enriched in the category of categories.

(ii) Characterize categories sufficiently like the category of sets. The reviewer has had his say about (i) in [8 and 9]. Further work has been carried out by R. Street; e.g., in [24 and 25]. Topic (ii) led to the third main direction away from set theory.

3. Topos theory. The next step after finding an elementary theory of the category of sets seems by hindsight obvious. Giraud in [26] had given a (nonelementary) characterization of the category of sheaves on a site. Such a category was called a topos by Grothendieck. Throughout the whole period considered here, categories of sheaves played a crucial role in guiding developments in category theory and in facilitating the extension of categorical

methods to other branches of mathematics. (See the reviewers article [10] for some of the history of this situation.) Sheaves of groups or modules were familiar things; perhaps sheaves were sets in some appropriate sense. In 1970–71, Lawvere and Tierney introduced the notion of an elementary topos as a particular kind of category. (A complete elementary topos is a Grothendieck topos.) It turned out a few years later than an elementary topos is precisely a model of higher order intuitionistic set theory, and thus is sufficiently like the category of sets for many purposes. For details see the books by Johnstone [11] and Goldblatt [7] as well as the many papers that have been published in the last decade.

Let us now turn to the book under consideration by imagining an ideal reader who has gone through a first presentation of category theory such as Mac Lane [19] and who would like to learn more. Not feeling ready to tackle research papers, he looks around for a more advanced book about categories. In topos theory—if the reader knows that's what is wanted—Johnstone or Goldblatt is there to be found. Other books that might turn up are Makkai and Reyes [20] or Koch [15] (an exciting, beautiful book about a new subject). Concerning 2-categories (categories enriched in the category of categories) [9] is not very suitable for a beginner; perhaps [14] is better. As to closed categories, up to now there was only Dubuc [3] in addition to the standard [5]. The amount of time people have spent struggling with the various axiom systems in [5] must be astronomical. Now, at least Kelly's new book is a candidate for a different approach. Three questions—prompted by Kelly's claim to have written a textbook—are relevant:

(i) Will our reader fully understand the contents of this book?

(ii) Will their significance be appreciated?

(iii) Will the reader be brought to the level where entry into the mainstream of current research is possible?

An honest answer to such grandiose questions has to be "no" for almost any book, but by treating the questions as goals we can ask what progress is made toward achieving them.

First of all, briefly, what is to be found in the book? The titles of the chapters are: 1. "The Elementary Notions"; 2. "Functor Categories"; 3. "Indexed Limits and Colimits"; 4. "Kan Extensions"; 5. "Density"; 6. "Essentially-Algebraic Theories Defined by Reguli and Sketches". The first four cover the basic core of the subject. One can argue about 5, but 6 is clearly a more special subject. The heart of our reader's problem is to get through Chapter one. The reason for this difficulty can be explained as follows: Many people are now familiar with the description of things like groups via commutative diagrams rather than equations. In order to treat the subject of closed categories, one must similarly describe categories, functors, and natural transformations by commutative diagrams. This is not intrinsically difficult but two nuisance technical problems crop up. One is called the coherence problem. Its depth is indicated by its relation to similar problems in logic and the fact that there is still work to be done on it. Kelly handles it by simply referring to the relevant literature and literally never mentioning it again although a full account of almost any proof in the subject uses coherence. The second problem

is that the appropriate notion of naturality, called V-naturality, is much harder to treat than ordinary naturality. It is often fairly easy to see how to generalize the usual constructions of category theory to the closed situation; what is not so easy is to prove that these constructions are V-natural. By the excluded middle, there are just two techniques for doing this: either make the relevant diagrams which are sometimes very large, or don't make them. Kelly has chosen the latter course and merely gives brief hints as to why certain statements are true. Thus, as with much of the current topos theory literature, there is a secret text behind the overt text in which the subject has a completely different, more detailed but intrinsically simpler appearance. In these circumstances, as well as in many other places in mathematics, the actual text functions as a metaphor for the hidden text, and the task of extracting the real meaning from the written word may in fact be impossible.

Even more drastically, Kelly has opted for a discursive style in the first two and a half chapters, eschewing any use of the standard definition-theorem-proof format until the middle of Chapter three. One of the main advantages of the standard format is that it provides a clear demarcation between places where mathematics is being done and places where we are being told what is going on. Kelly's informal presentation provides no such demarcation and it is some time before we realize that we are almost never told what is going on—that almost every line is making a mathematical assertion whose proof requires our participation together with a page or two of scribbling. A vast amount of information is summarized in about 60 pages this way. There are indications of proofs, although far too many things are regarded as "being easily seen". What I think these proofs amount to is the following: if our reader constructs the requisite large diagram, then the commutativity of certain subregions will not be immediately evident. Clues are given as to why these regions commute. Since nowhere is even a simple proof worked out, I don't see how our reader will realize that this is what he must do, right from the very beginning, without skipping anything. Furthermore, if a professional drops in on the middle of the book to find out how some particular result is proved, he will find himself in an almost infinite regress of such "proof suggestions". (Cf. Bonsall [1].) It seems that our reader will have no recourse but to turn to Eilenberg and Kelly [5]. If he provides himself with a table of contents for that paper he will find exactly the kind of organization that is missing from this book. Furthermore, he will find a fairly extensive treatment of examples, something that is also absent from this book. Unfortunately, that will only get him through Chapter one.

At this point it seems wise to drop the pretense—suggested by the author— that this is a textbook. It is not; it is one man's account of how he sees a large portion of category theory, both in the set-based and V-based cases, founded on his 20 years of experience and work in the field. The account is interesting and useful and filled with many insights. Kelly is the master of this field and is the originator of most of the central concepts. He is solely responsible for the main technique, that of describing Kan extensions, indexed limits, and enriched homs in functor categories by end formulas and then using the preservation of ends by representable functors, the ends formulation of the Yoneda

theorem, and the Fubini Theorem to get further results. This book will be the standard reference in the subject, presumably for a long time, just because nobody else has a clear idea how to present the basic ideas of the subject, but it is not for neophytes.

Chapters two, three, and four constitute the high point of the book. They are based on [2, 12, and 3], and provide a fuller account of this material than has hitherto been available. Given the material of Chapter one, they are somewhat easier sailing. The standard material is there together with a number of important distinctions between ordinary category theory and closed category theory. For instance, there is a careful discussion of the relation between ordinary limits and indexed limits as well as the relation between the universal mapping property of Kan extensions and the usual formula for such extensions. However, as in Chapter one, proofs of V-naturality of things like the Fubini isomorphism, the Yoneda isomorphism, and various aspects of Kan extensions seem excessively brief and off handed. Presumably these results and techniques are what the rest of the world needs to know about working with closed categories. Perhaps detailed proofs are not so important, but precise statements should be clearly formulated and suitably highlighted to make them easy to locate and refer to. It cannot be said that this has been done here.

On the other hand, parts of Chapters three and four together with all of Chapters five and six contain the material which is clearly of greatest interest to Kelly. This material, presented in the usual theorem-proof format, constitutes a research monograph developing background material for a study of a very general version of enriched algebraic categories. The material is relentlessly abstract and, as far as I can see, suitable only for specialists who will themselves profit from looking first at [13] to get some idea of the intended use of these chapters. Even in [13], which is closely related to this book and does contain many examples, one feels the lack of the detailed working-out of some particular example where the force of the many theorems would play a role. The kind of example that is needed is one in which the base category V is neither cartesian closed nor R-modules (or differential graded R-modules) since special methods adapted to these cases seem simpler to use than the general theory. The only example that comes to mind is that of Banach spaces. So I close with a question: what are the interesting Ban-enriched categories and how does this theory apply to them. For a start on answers, see [22 and 23].

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The evolution of dynamics, vibration theory from 1687 *to* 1742, by John T. Cannon and Sigalia Dostrovsky, Studies in the History of Mathematics and Physical Sciences, Vol. 6, Springer-Verlag, New York, 1981, ix + 184 pp., $38.00. ISBN 0-3879-0626-6

The publication in 1687 of Isaac Newton's monumental treatise *Principia Mathematica* has long been regarded as the event that ushered in the modern period in mathematical physics. Newton developed a set of techniques and methods based on a geometric form of the differential and integral calculus for dealing with point-mass dynamics; he further showed how the results obtained could be applied to the motion of the solar system. Other topics studied in the