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Linear orderings, by Joseph G. Rosenstein, Academic Press, New York, 1982, xvii + 487 pp., $64.00. ISBN 0-1259-7680-1

The study of linear (total) orderings requires no justification. The concept is one of the earliest and most fundamental in mathematics—its key importance was established by such giants as Cantor, Gödel and von Neumann who pinpointed the central role of the ordinal numbers. The complexity of more general linear orderings was shown by Hausdorff in "Grundzüge einer Theorie der geordneten Mengen" which laid the foundations of the subject. More recently, developments in universal algebra and model theory have led to renewed interest in linear orderings. In 1977, Graham Higman proved that any homogeneous relation is essentially a linear ordering. In model theory, Ehrenfeucht and Mostowski (1956) showed that if a first order theory T has an infinite model and $\langle X, \leqslant \rangle$ is any linearly ordered set, then T has a model whose automorphism group has $\mathrm{Aut}(\langle X, \leqslant \rangle)$ as a subgroup. In the 1970s, Shelah developed a technique (forking) for analysing first order theories in which no infinite linear ordering is implicitly defined (stable theories); no such general technique is known for handling theories in which an infinite linear ordering is present.

In the absence of algebraic operations and any other relations, theories of linear orderings are quite well understood, mainly as a result of Laüchli and Leonard, Rosenstein, Rubin, Gurevich and Shelah, and Fraïssé and his school. For example, Rubin proved that any complete theory of linear orderings which has an uncountable number of countable models has continuum many—a very deep theorem. Some of the work has been generalised to partial orderings, but the absence of any overall picture when algebraic operations interplay with the ordering remains acute. Isolated examples have been extensively studied:

models of arithmetic and real closed fields (with considerable overlap with number theory), lattice-ordered rings, especially *f*-rings (origins in the great work of the Rieszes in functional analysis), ordered abelian groups and lattice-ordered groups (essentially algebra). I personally believe that generalising these widely discrepant disciplines into a coherent picture, though extremely difficult, should be the most immediate goal of abstract model theory. Since every lattice-ordered group can be embedded in Aut($\langle X, \leqslant \rangle$) with the pointwise ordering for some linearly ordered set $\langle X, \leqslant \rangle$.(W. Charles Holland), this obverse side of the Ehrenfeucht-Mostowski coin suggests that the theory of lattice-ordered groups may play the same pivotal role in the study of unstable theories that fields played in the study of stable ones. Moreover, knowledge of permutation groups and linear orderings make this study more tractable—e.g., to prove the undecidability of the theory of lattice-ordered groups, the existence of a 4 generator 1 relator lattice-ordered group with insoluble (group) word problem, and the undecidability of the isomorphism and triviality problems for finitely presented lattice-ordered groups.

Having delineated a general direction I would like a branch of mathematics to go in and explained my own more narrow interest (lattice-ordered groups), it is time for me to survey the great progress made in the study of linear orderings. The theory of scattered linear orderings (i.e., those in which the rational line cannot be embedded) is well understood due to Hausdorff. A set of finitely axiomatisable linear orderings giving a representative for each equivalence class of linear orderings under a weak elementary equivalence was obtained by Laüchli and Leonard even when the linear ordering is not necessarily scattered. Certain strange subsets of the real line are known to exist, e.g., there is an uncountable subordering of the real line which is embeddable in every uncountable subordering of the real line—in some model of set theory. There are also several partition theorems of a combinatorial nature for order types. Fraïssé's conjecture that there is no infinite descending sequence of countable order types was established by Laver using deep combinatorial theorems of Nash-Williams, as were several others of a more technical nature. Rubin's result referred to above (and generalised to complete theories of certain trees by Steel) which grew out of a detailed study of the 2-types of complete theories of linear orderings and Rosenstein's characterisation of $\aleph_0$-categorical complete theories of linear orderings cover most of the main first order results. Landraitis proved that those countable linear orderings which are completely characterised among *all* linear orderings by a statement of the infinitary language $L_{\omega_1\omega}$ are precisely those in which every orbit is scattered. Rubin also generalised his own theorem to this more general language. In the case of monadic second order logic, only partial results are obtainable, the dependence on set theory being crucial. Surprisingly, the weak second order theory of linear orderings is decidable—independent of set theory (Laüchli)—as is the second order theory of the binary branching tree with direction (Rabin). Finally, there has been considerable work done on recursive linear orderings and generalisations, mainly by Lerman and Schmerl. All of these topics (in the linearly ordered case) are covered carefully and fully in Rosenstein's book.

Although any two dense linearly ordered sets are indistinguishable by first order sentences and only partly distinguishable by monadic properties (Gurevich, Magidor and Shelah), some resolution is possible if the perspective is slightly changed and automorphism groups are considered. For example, there is a first order sentence of the language of group theory that is satisfied by the automorphism group of a 1-homogeneous linearly ordered set if and only if the linearly ordered set is the real line (Gurevich, Holland, Jambu-Giraudet and Glass)—and this does not depend on set theory, nor do similar results.

Like this review, the book has a decidedly model-theoretic slant and relies heavily on the Ehrenfeucht-Fraïssé game technique, which I found helpful. Anyone who has taught a course in model theory knows only too well how difficult it is to find theories in which students can "get their hands dirty" and perform detailed computations. Linear orderings provide one of the few exceptions so it is very nice to see that Rosenstein has peppered his pages with calculations. This is further brought out by many exercises—4.4.4.(3) is most challenging; it contradicts 4.4.4.(5) and is one of the few misprints I found (the others are obvious). The book has clearly been written with the graduate student in mind; I have never read a mathematics book before where the author has so obviously chosen his words carefully to ensure that the reader is in the right frame of mind when he or she comes across each new concept. It can certainly be used for a graduate course in infinite combinatorics or in model theory; in the latter case, it might be used as supplementary reading for such a course, though since all the standard model theory is introduced thoroughly, with proofs, before linear orderings are considered specifically, this is not necessary. However, there is such a mine of information in the book that it will be useful for researchers in these disciplines as well as other mathematicians and is a must for libraries. Because of the price, I fear that they and reviewers will be the only owners! Its appearance now that there is a need to study unstable theories is most propitious. The only complaints I have are that Rosenstein makes no attempt in his book, even in his introduction, to relate his subject to other areas of mathematics—something I have tried to right in this perhaps lopsided review—nor, where possible, to deal with partial orderings when little or no extra work is needed. However, any complaints are most churlish since I learnt a lot from the book, have great admiration for the author's own central work and influential role in the subject, and found this a superbly written book.

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Hankel operators on Hilbert space, by S. C. Power, Research Notes in Mathematics, No. 64, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1982, 87 pp., $13.95. ISBN 0-273-08518-2

Hankel operators have been studied on and off for years, because they arise in a variety of problems of complex analysis and operator theory. At their