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Additional physical insight can be gained from the many excellent articles on the subject which have appeared over the past decade in Scientific American (articles by Glashow, Nambu and others).

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

0273-0979/83 $1.00 + $.25 per page

Lectures on Riemann surfaces, by Otto Forster, Graduate Texts in Math., vol. 81, Springer-Verlag, New York, 1981, viii + 254 pp., $35.00. ISBN 0-3879-0617-7

Riemann surfaces, By Hershel M. Farkas and Irwin Kra, Graduate Texts in Math., vol. 71, Springer-Verlag, New York, 1981, x + 337 pp., $29.80. ISBN 0-3879-0465-4

Riemann surfaces, those old and venerated structures, show their smiling faces in many different connections, from the geometry of algebraic curves to the integration of nonlinear partial differential equations in mathematical physics. Even with all that is familiar, each generation finds frontiers beyond, exciting the explorers with a unique combination of explicitness and richness of technique: algebraic, analytic, and geometric. The great masters of the 19th century (Abel, Riemann,...) left a wealth of information and insight on algebraic and abelian functions. Around the turn of the century the uniformization theorem became firmly established by Poincaré and Koebe. (As Ahlfors says in [**2**], "This is perhaps the single most important theorem in the whole theory of analytic functions of one variable.") At about the same time, the theory of fuchsian groups and automorphic functions, which through the uniformization theorem provides its own approach to the subject of Riemann surfaces, was formulated by Poincaré. The classic book of Hermann Weyl in 1913 put the abstract definition of Riemann surface on a solid foundation (still an excellent introduction in the English translation of its third (1955) edition). Additional fundamental contributions were made by the towering figures, Nevanlinna, Teichmüller and Ahlfors. Some of the most active areas of research in recent years concern the Teichmüller theory and its ramifications, including measured foliations and the geometry and dynamics of automorphisms, the general Schottky problem of characterizing jacobian period matrices, and the spectral decomposition of the Laplace-Beltrami operator.

These remarks can hardly be taken as a history but at least they establish our point of view, which is analytic rather than algebraic. Perhaps it is helpful to classify the subject as follows:

A. The algebraic theory: compact Riemann surfaces without boundary as complex algebraic curves (see Mumford's lectures [**8**] or the Griffiths and Harris book [**4**]).

B. The analytic theory: construction of meromorphic functions and differentials by analytic methods; noncompact surfaces and the uniformization theorem.

Under this second heading it is convenient to make the following classification, even though the three divisions interact quite beautifully.

1. The theory of compact surfaces.

2. The theory of noncompact surfaces: surfaces that may have infinite genus and/or infinitely many ideal boundary components. (For an interesting history, see Ahlfors' lecture [**3**].)

3. The theory of fuchsian groups.

The two books under review are introductions to the subject from the point of view of (1) and (2). The Farkas-Kra work adopts the classical approach and treats in detail compact surfaces while the Forster book uses the tools developed for complex manifold theory in higher dimensions to give a broad based introduction without putting special emphasis on any one area. Unfortunately the books do not contain "dictionaries" spelling out the interconnections between the two approaches and the algebraic one. We begin with the lists of chapter headings.

The Farkas-Kra book has the following chapters: O. An Overview; I. Riemann Surfaces; II. Existence Theorems; III. Compact Riemann Surfaces; IV. Uniformization; V. Automorphisms of Compact Surfaces—Elementary Theory; VI. Theta Functions; VII. Examples.

The Forster book is divided as follows: 1. Covering Spaces; 2. Compact Riemann Surfaces; 3. Non-compact Riemann Surfaces.

We will briefly discuss the two in a number of specific areas, comparing them to each other and to some other general texts available in English.

Basic topology. This is covered in more detail in the Forster book, especially the fundamental group and the important matter of the construction of covering surfaces. Homology is approached via cohomology and the Riemann-Hurwitz formula, the basic relation between the Euler characteristics of the covering and base surfaces, is derived from the Serre Duality Theorem (!). The Farkas-Kra book contains only a brief review but that does include the simple proof of the Riemann-Hurwitz formula (later, it is also derived from the Riemann-Roch theorem). More insight is provided much later in the book during the construction of branched, simply connected coverings by fuchsian groups. In neither book is there a discussion of doubling, or of the special topology of noncompact surfaces. Ahlfors and Sairo **[1]** is still the most complete source for basic surface topology, although that too needs to be supplemented especially in the direction of mappings of surfaces.

Definition of Riemann surface. In both books it is defined straight away as a one-dimensional complex manifold via coordinate charts. What if a student has no "feel" for this abstract definition? Farkas and Kra are satisfied with the example of the Riemann sphere early on—and suggest others in the preliminary introduction—but even the torus from a plane lattice is not fully discussed until p. 212. Forster sets the torus example as an early exercise and puts a discussion of how Riemann surfaces arise from algebraic functions after introducing sheaves in Chapter 1. (These are the *original* surfaces of Riemann, although Riemann's construction with cross-identified slits between branch points is never mentioned in one book and barely mentioned in the other.) In Farkas-Kra, the discussion of algebraic surfaces is put off until p. 226. Neither book suggests, for example, how the notion of Riemann surface is very useful for geometrically analyzing and even constructing meromorphic functions. Neither book discusses the notion of isothermal coordinates which not only is the natural approach to turning surfaces embedded in 3-space into Riemann surfaces, but also forms the basis for complex deformation theory. Perhaps best in the motivation category are Siegel's volumes 1 and 2 **[9]**, George Springer's widely used and very readable text **[10]**, and the classic **[11]**.

Construction of analytic and harmonic forms. Traditionally the methods are (i) the solution of the Dirichlet problem together with devices to extend local solutions globally via exhaustions, and (ii) the projection from various classes of square integrable forms to harmonic ones using Weyl's lemma. As in Ahlfors-Sario **[1]** (where (i) is neatly formulated in terms of "principal operators"), Farkas and Kra develop both. A nice feature is their demonstration that each in its own way can be used to derive the Riemann-Roch theorem. Their proof of the uniformization theorem is based on (i) which yields Green's function when it exists. Forster brings in Dolbeault's lemma (local solution of $\bar{\partial} f = g$) together with sheaf cohomology and Serre duality. That much suffices for compact surfaces. Added for noncompact ones, are the Dirichlet problem and Weyl's lemma. These are used to give a proof of the Runge theorem of Behnke and Stein (which can also be proved in the classical manner using a Cauchy integral formula on compact Riemann surfaces with boundary). That in turn is used to prove the uniformization theorem.

Compact surfaces. While the Forster book sticks to the basics, concluding with the Jacobi inversion theorem, it is here that the Farkas-Kra book really shines. The subject is penetrated rather deeply, to my knowledge the only text using the classical approach which does so. There are nice treatments of Abel's theorem and Jacobi inversion, the Noether gap theorem, and the Weierstrass points. The varieties of special divisors in the jacobian are discussed in some detail including a proof of M. Noether's theorem on spanning holomorphic q-differentials by abelian (first order) ones. A number of other topics are gone into as well. The most extensive of these is the discussion of automorphisms comprising Chapter V. This reinforces our understanding of Chapter III and displays some interesting inner algebraic properties. The introduction to theta functions and proof of Riemann's vanishing theorem contained in Chapter VI is very useful too. Actually these chapters can be used for a self-contained course on compact surfaces; the uniformization theorem is not needed (that in turn can be presented separately as in [2]). But their fans want even more for the second edition: a proof of Torelli's theorem that surfaces with the same period matrices are conformally equivalent, and more worked out examples in low dimensions than what appears in Chapter VII.

Noncompact surfaces. In Farkas-Kra there is not much beyond what is required for the uniformization theorem. Forster's book contains more. The Runge theorem is applied to obtain both "Mittag-Leffler" and "Weierstrass" developments for meromorphic functions. This work continues toward the proof that holomorphic line bundles on open surfaces are holomorphically trivial. The book ends with Röhrl's solution of the Riemann-Hilbert problem. Much earlier in the book, there is a nice introduction to linear differential equations.

Personally, I favor the book of Ahlfors and Sario [**1**] as a general introduction to Riemann surfaces. Yet that needs to be substantially supplemented. In one direction there is very little on compact surfaces. Here the Farkas-Kra book comes to the fore. In my opinion it is worth the price just for those excellent chapters; they have served me well for some years both as a text and as a reference. Neither does Ahlfors-Sario contain the important Runge theorem and its consequences. Furthermore, since it was written techniques from several complex variables and algebraic geometry have very much come into the limelight. For compact surfaces, introductions to those methods can be found in the book of Griffiths-Harris [**4**] and especially in Gunning's comprehensive trilogy [**5—7**]. Forster's book provides excellent preparation for study in this direction. Pedagogically it is very systematic, proceeding leisurely with illustrative exercises at the end of every section and carefully planned goals. Farkas and Kra have written a looser book, rich with asides, parenthetical remarks and supplementary information. I recommend both very highly. Together they exhibit the breadth of the theory of functions of one complex variable: Limited in the domain but unlimited in the range!!

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

0273-0979/83 $1.00 + $.25 per page

Linear orderings, by Joseph G. Rosenstein, Academic Press, New York, 1982, xvii + 487 pp., $64.00. ISBN 0-1259-7680-1

The study of linear (total) orderings requires no justification. The concept is one of the earliest and most fundamental in mathematics—its key importance was established by such giants as Cantor, Gödel and von Neumann who pinpointed the central role of the ordinal numbers. The complexity of more general linear orderings was shown by Hausdorff in "Grundzüge einer Theorie der geordneten Mengen" which laid the foundations of the subject. More recently, developments in universal algebra and model theory have led to renewed interest in linear orderings. In 1977, Graham Higman proved that any homogeneous relation is essentially a linear ordering. In model theory, Ehrenfeucht and Mostowski (1956) showed that if a first order theory T has an infinite model and $\langle X, \leqslant \rangle$ is any linearly ordered set, then T has a model whose automorphism group has $\mathrm{Aut}(\langle X, \leqslant \rangle)$ as a subgroup. In the 1970s, Shelah developed a technique (forking) for analysing first order theories in which no infinite linear ordering is implicitly defined (stable theories); no such general technique is known for handling theories in which an infinite linear ordering is present.

In the absence of algebraic operations and any other relations, theories of linear orderings are quite well understood, mainly as a result of Laüchli and Leonard, Rosenstein, Rubin, Gurevich and Shelah, and Fraïssé and his school. For example, Rubin proved that any complete theory of linear orderings which has an uncountable number of countable models has continuum many—a very deep theorem. Some of the work has been generalised to partial orderings, but the absence of any overall picture when algebraic operations interplay with the ordering remains acute. Isolated examples have been extensively studied: