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THE SULLIVAN CONJECTURE

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THEOREM 0. *Let G be a locally finite group with classifying space BG and let X be a connected finite dimensional CW complex. Then the space of pointed maps from BG to X is weakly contractible:* $\pi_i(\mathrm{map}_*(BG,X)) = 0$ *for all $i \geq 0$. In particular, every map from BG to X is null-homotopic.*

A group is locally finite if it is a direct limit of finite groups. When G is of order 2, this theorem represents an affirmative resolution of part of a conjecture of Dennis Sullivan [**11**]. In this announcement I will sketch a proof of this theorem; details will appear in due course.

Theorem 0 is a straightforward consequence of the following three results.

THEOREM 1. *If X is a connected CW complex of finite category and G is a torsion group, then every map $f\colon BG \to X$ is trivial in π_1.*

THEOREM 2. *If X is a nilpotent space such that $H_*(X;\mathbf{F}_p)$ is bounded—that is, $H_i(X;\mathbf{F}_p) = 0$ for all large i—then* $\mathrm{map}_*(BZ_p,X)$ *is weakly contractible.*

THEOREM 3. *Let X be a nilpotent space and G a locally finite group. If* $\mathrm{map}_*(BZ_p,X)$ *is weakly contractible for every prime p occurring as the order of an element of G, then* $\mathrm{map}_*(BG,X)$ *is weakly contractible.*

Theorem 1 is an exercise in K-theory and covering spaces. For the remaining theorems, it is convenient to work simplicially; the topological results follow from standard comparison theorems. So "space" will always mean "simplicial set", usually assumed fibrant.

PROOF OF THEOREM 2. The crux is the following theorem, which distills results of Bousfield and Kan [**5**], and Dror, Dwyer, and Kan [**7**].

THEOREM 2.1. *Let W be a connected space such that $\overline{H}_*(W;\mathbf{Z}[\frac{1}{p}]) = 0$, and let X be nilpotent. Then the natural map $X \to \mathbf{F}_{p\infty}X$ to the Bousfield-Kan $\mathbf{F}_p$-completion induces a weak equivalence*

$$\mathrm{map}_*(W,X) \to \mathrm{map}_*(W,\mathbf{F}_{p\infty}X).$$

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Let $\mathcal{C}$ be the category [4] of unstable right module coalgebras without unit over the mod p Steenrod algebra A. Then [4 and 5] show that

$$\mathrm{map}_*(W, \mathbf{F}_{p\infty}X) \simeq *$$

provided that

$$\mathrm{Ext}^s_{\mathcal{C}}(\overline{H}_*(\Sigma^t W), \overline{H}_*(X)) = 0 \quad \text{for all } t \geq s \geq 0.$$

Here $\mathrm{Ext}^s_{\mathcal{C}}(\overline{H}_*(\Sigma^t W), -)$ is the sth cosimplicially derived functor of $\mathrm{Hom}_{\mathcal{C}}(\overline{H}_*(\Sigma^t W), -)$. When $t = 0$, only $\mathrm{Ext}^0_{\mathcal{C}}(\overline{H}_*(\Sigma^t W), -)$ is defined, and it is only a pointed set. When $t > 0$, $\overline{H}_*(\Sigma^t W)$ is an Abelian cogroup in $\mathcal{C}$ and so $\mathrm{Ext}^s_{\mathcal{C}}(\overline{H}_*(\Sigma^t W), -)$ is defined for all $s \geq 0$ and is an $\mathbf{F}_p$-vector space. Theorem 2 follows from (2.1) and

THEOREM 2.2. *If $C \in \mathcal{C}$ is bounded then*

$$\mathrm{Ext}^s_{\mathcal{C}}(\overline{H}_*(\Sigma^t BZ_p), C) = 0 \quad \textit{for all } t \geq s \geq 0.$$

The case $t = 0$ of this theorem follows from the fact that every element of $\overline{H}_*(BZ_p)$ is the image of a positive dimensional Steenrod operation. For the remaining cases we invoke a composite functor spectral sequence. Let N be any object of the category $\mathcal{U}$ of unstable right A-modules. Then $\Sigma N \in \mathcal{C}$, with trivial diagonal. The module $PC = \ker(\Delta\colon C \to C \otimes C)$ of primitives is a suspension in $\mathcal{U}$, and $\mathrm{Hom}_{\mathcal{C}}(\Sigma N, C) \cong \mathrm{Hom}_{\mathcal{U}}(N, \Sigma^{-1} PC)$. We obtain a convergent spectral sequence

$$\mathrm{Ext}^s_{\mathcal{U}}(N, \Sigma^{-1} R^t P(C)) \Rightarrow \mathrm{Ext}^{s+t}_{\mathcal{C}}(\Sigma N, C).$$

Theorem 2.2 therefore follows from the next two results.

THEOREM 2.3. *If C is a connected bounded $\mathbf{F}_p$-coalgebra then $R^t P(C)$ is bounded for each t.*

THEOREM 2.4. *If $M \in \mathcal{U}$ is bounded then $\mathrm{Ext}^s_{\mathcal{U}}(\overline{H}_*(\Sigma^n BZ_p), M) = 0$ for all s, $n \geq 0$.*

PROOF OF THEOREM 2.3. Since any connected coalgebra is a direct limit of finite coalgebras, we may prove (2.3) by establishing suitable results for left derived functors of the functor assigning to a connected commutative $\mathbf{F}_p$-algebra R (henceforth, "algebra") its module of indecomposables QR. As a first approximation we have the following theorem of André [1]:

THEOREM 2.3.1. *If QR is finite, then $L_t Q(R)$ is finite for each t.*

André's proof does not seem to yield a bound on the top nonzero dimension, however, so sharper methods are needed to establish (2.3) in general. Quillen [10] tells us to think of $L_* Q(R)$ as the "homology" of the algebra R; more generally, any simplicial algebra X admits a surjective weak equivalence $\tilde{X} \to X$ from a cofibrant object [10], and $\pi_*(Q\tilde{X})$ is the "homology" of X. Quillen constructs an "Adams spectral sequence", passing from $\pi_*(Q\tilde{X})$ to $\pi_*(X)$, and Bousfield [3] carries out the coalgebra analogue. We construct a "reverse

Adams spectral sequence". This spectral sequence is uninteresting when applied to a constant simplicial algebra R. However, "homology commutes with suspension", and in this setting, the Eilenberg-Mac Lane functor $\bar{W}$ is suspension [9]. Let D be the algebra of higher divided powers, studied at the prime 2 by Dwyer [8] and in general by Bousfield in earlier but unpublished work [2]. D acts naturally on $Q\pi_*(X)$ for any simplicial algebra X—in particular, on the vector space of indecomposables in $\pi_*(\bar{W}R) = \mathrm{Tor}_*^R(\mathbf{F}_p, \mathbf{F}_p)$—and satisfies a certain "unstable" condition. Let $\mathrm{Untor}_*^D(\mathbf{F}_p, -)$ denote the derived functors of $\mathbf{F}_p \otimes_D$—on the category of unstable D-modules. The spectral sequence then takes the following form.

THEOREM 2.3.2. *There is a natural spectral sequence*

$$E^2_{s,t} = \mathrm{Untor}^D_{s,t}(\mathbf{F}_p, Q\ \mathrm{Tor}_*^R(\mathbf{F}_p, \mathbf{F}_p)) \Rightarrow L_{s+t-1}Q(R).$$

The boundedness result (2.3) follows easily, as does (2.3.1). One also obtains as a corollary a case of an unpublished result of Quillen, namely, that

$$Q\ \mathrm{Tor}_n^R(\mathbf{F}_p, \mathbf{F}_p) \xrightarrow{\cong} L_{n-1}Q(R) \quad \text{for } 1 \leq n \leq 2p-1.$$

PROOF OF THEOREM 2.4. An easy spectral sequence argument shows that we may assume $n = 0$. Let $G(n)$, $x_n \in G(n)_n$, corepresent the functor from $\mathcal{U}$ to sets given by $M \mapsto M_n$: that is, $\mathrm{Hom}_A(G(n), M) \xrightarrow{\cong} M_n$ by $f \mapsto f(x_n)$. $G(n)$ is evidently projective in $\mathcal{U}$. We have maps $G(2n) \to G(2pn)$ by $x_{2n} \mapsto x_{2pn}P^n$ (where $P^n = Sq^{2n}$ if $p = 2$). Write G_{2n} for $\varinjlim G(2p^i n)$. There are natural maps $G_{2n} \to \overline{H}_*(BZ_p)$. Theorem 2.4 for $n = 0$ is a direct consequence of the next theorem, due when $p = 2$ to Carlsson [6].

THEOREM 2.4.1. $\bigoplus_{n=1}^{p-1} G_{2n} \to \overline{H}_*(BZ_p)$ *is a split epimorphism.*

PROOF OF THEOREM 3. Let G be a group, and consider a subcategory $\mathcal{O}$ of $\mathcal{O}(G)$, the category of transitive G-sets and equivariant maps, satisfying

(*) $\mathcal{O}$ is a full subcategory containing an orbit T_0 with a point x_0 whose isotropy subgroup fixes every orbit in $\mathcal{O}$.

Examples of interest include: (i) Let H be any subgroup of G and $\mathcal{O}_H$ the category of all those orbits whose isotropy groups are intersections of conjugates of H; (ii) Let p be a prime and $\mathcal{O}_p(G)$ the category of all orbits whose isotropy groups are p-groups. Let $T\colon \mathcal{O} \to (G\text{-sets})$ be the inclusion functor. Let hocolim be the $\underrightarrow{\mathrm{holim}}$ of [5].

THEOREM 3.1. *Let $\mathcal{O}$ satisfy (*) and let E be a principal G-space. There is a natural homotopy equivalence*

$$\operatorname*{hocolim}_{\mathcal{O}} E \times_G T \simeq E/G.$$

In particular, if E is contractible then $E \times_G T \simeq BH$ where H is an isotropy group of T. We then take the liberty of writing (3.1) as

$$\operatorname*{hocolim}_{\mathcal{O}} BH \simeq BG.$$

The resulting spectral sequence [**5**], for an additive homology theory h_*,

$$L_* \operatorname*{colim}_{\mathcal{O}} h_*(BH) \Rightarrow h_*(BG),$$

would appear to be a useful tool in the study of group cohomology, unifying several well-known constructions.

Theorem 3 results from (3.1), the arithmetic square of [**7**], and the following consequence of the Sylow theorem.

THEOREM 3.2. *Let $G(p)$ be a p-Sylow subgroup of the finite group G. Then $\overline{H}_*(\mathcal{O}; \mathbf{Z}_{(p)}) = 0$, where $\mathcal{O}$ is (the nerve of) either $\mathcal{O}_p(G)$ or $\mathcal{O}_{G(p)}$.*

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REFERENCES

1. M. André, *Homologie des algèbras commutatives*, Grundlehren der Math. Wissenschaften, Band 206, Springer-Verlag, Berlin and New York, 1974.
2. A. K. Bousfield, *Operations on derived functors of nonadditive functors*, manuscript.
3. ——, *Nice homology coalgebras*, Trans. Amer. Math. Soc. **148** (1970), 473–489.
4. A. K. Bousfield and D. M. Kan, *The homotopy spectral sequence of a space with coefficients in a ring*, Topology **11** (1972), 79–106.
5. ——, *Homotopy limits, completions, and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin and New York, 1972.
6. G. Carlsson, *G. B. Segal's Burnside ring conjecture for* $(\mathbf{Z}/2)^k$, Topology **22** (1983), 83–103.
7. E. Dror, W. G. Dwyer, and D. M. Kan, *An arithmetic square for virtually nilpotent spaces*, Illinois J. Math. **21** (1977), 242–254.
8. W. G. Dwyer, *Homotopy operations for simplicial commutative algebras*, Trans. Amer. Math. Soc. **260** (1980), 421–435.
9. S. Mac Lane, *Constructions simpliciales acycliques*, Colloque Henri Poincaré, 1954.
10. D. G. Quillen, *On the homology of commutative rings*, Proc. Sympos. Pure Math., vol. 17, Amer. Math. Soc., Providence, R.I. 1980, pp. 65–87.
11. D. Sullivan, *Geometric topology, Part* I: *Localization, periodicity, and Galois symmetry*, M.I.T., Cambridge, Mass., 1970.

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