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Functional integration and quantum physics, by Barry Simon, Academic Press, New York, 1979, ix + 296 pp., $29.50.

Quantum physics, a functional integral point of view, by James Glimm and Arthur Jaffe, Springer-Verlag, New York, 1981, xx + 417 pp., $16.80.

These two books have strikingly similar titles, but the topics discussed are almost disjoint. Common to both is the approach to problems of quantum physics by first studying them in imaginary time.

If we know all about the selfadjoint operator H, then we know all about the solution $\psi = e^{-itH}\psi_0$ of the Schrödinger equation; but if we know all about $f = e^{-tH}f_0$ (the solution of the Schrödinger equation in "imaginary time"), then we know all about H.

The simplest H of interest in quantum physics is of the form $-\frac{1}{2}\Delta + V$ on $L^2(\mathbf{R}^\nu)$, where V is the operator of multiplcation by a function. This case is discussed in depth by Simon. The Feynman-Kac formula gives an explicit expression for the kernel of the integral operator e^{-tH}.

$$e^{-tH}(a, b) = \int \exp\left(-\int_0^t V(\omega(s))\, ds\right) d\mu_{0,a,b;t}. \tag{1}$$

The integration is over the space of all paths ω: $\mathbf{R} \to \mathbf{R}^\nu$ and $\mu_{0,a,b;t}$ is the condition Wiener measure for paths starting at a at time 0 and ending at b at time t.

As a simple example of the power of functional integration, Simon gives Symanzik's proof of the Golden-Thompson inequality

$$\operatorname{Tr} e^{-tH} \leqslant \int \frac{d^\nu p\, d^\nu x}{(2\pi)^\nu} e^{-t(p^2/2 + V(x))}. \tag{2}$$

The trace on the left contains a wealth of information on the distribution of eigenvalues; the integral on the right is a classical phase space integral. To evaluate the trace, set $a = b$ in (1) and integrate over $\mathbf{R}^\nu$. We can estimate

$$\exp\left(-\int_0^t V(\omega(s))\, ds\right) \leqslant \frac{1}{t}\int_0^t \exp(-tV(\omega(s)))\, ds,$$

by Jensen's inequality, and then (2) follows. Details are in Simon's Theorem 9.2; the point is that the usual tools of measure theory (monotone convergence, Jensen's inequality, etc.) can be brought to bear when working in imaginary time since we have a probability measure on paths. Simon discusses a vast number of other applications of functional integration to the Schrödinger equation. Of particular interest to probabilists, because of the occurrence of stochastic integrals, is the discussion of Schrödinger operators with magnetic fields.

Roughly speaking, Simon in this book discusses ν-dimensional random functions of 1 variable whereas Glimm and Jaffe discuss one-dimensional random functions of ν variables. The main topic of the Glimm-Jaffe book is constructive quantum field theory for a scalar field in two space-time dimensions, with an imaginary time approach. When the time in Minkowski space-time is made imaginary, what results in Euclidean space. The free Euclidean field is the Gaussian stochastic process ϕ indexed by test functions f, with mean 0 and covariance

$$E\phi(f)\phi(g) = \left\langle f, (-\Delta + m^2)^{-1} g \right\rangle,$$

where E denotes the expectation and m^2 is a positive constant. This process, invented by Loren Pitt, is a natural generalization to higher dimensions of the Wiener process. Call the corresponding probability measure μ_0; it may be realized on the space of distributions.

In a formal sense, the probability measure μ corresponding to a polynomial self-interaction P is given by

$$d\mu = \frac{1}{Z} \exp\left(-\int_{\mathbf{R}^2} P(\phi(x))\, dx \right) d\mu_0 \tag{3}$$

where Z is a normalization constant. There are some problems with this: (i) How does one form a polynomial in a distribution? (ii) Formal perturbation theory indicates that lower-order coefficients of P should be infinite. (iii) The measure μ_0 is ergodic under translations, so there is no translation-invariant probability measure μ that is absolutely continuous with respect to it other than μ_0 itself.

To approach these problems, replace $\mathbf{R}^2$ by the lattice $\varepsilon\mathbf{Z}^2$. We need estimates independent of ε so that we can take the continuum limit $\varepsilon \to 0$. Let Λ be a bounded region of $\mathbf{R}^2$, and let $\Lambda_\varepsilon = \Lambda \cap \varepsilon\mathbf{Z}^2$. We need estimates independent of Λ so that we can take the infinite volume limit $\Lambda \to \mathbf{R}^2$. In the colorful language of physics, these two problems are referred to as the ultraviolet problem and the infrared problem. Then the analogue of (3) is

$$d\mu_{\varepsilon\Lambda} = \frac{1}{Z_{\varepsilon\Lambda}} \exp\left(-\sum P(\phi(x))\varepsilon^2 - \frac{1}{2} \sum \phi(x) A_{\varepsilon\Lambda}(x, y) \phi(y) \right) \Pi\, d\phi(x),$$

where $A_{\varepsilon\Lambda}(x, y)$, for x and y in Λ_ε, is the matrix that is $4 + m^2\varepsilon^2$ for $x = y$, -1 for $|x - y| = \varepsilon$, and 0 elsewhere (this is the matrix of $-\Delta_{\varepsilon\Lambda} + m^2$, where $\Delta_{\varepsilon\Lambda}$ is the difference-operator analogue of the Laplacean). We can read off some properties of this measure. There is a weighting factor, involving the polynomial P, at each lattice site. Then to each pair of neighboring lattice sites x and y there is a term $\phi(x)\phi(y)$ in the exponent. Those configurations ϕ for which the exponent is large are favored by the probability measure $\mu_{\varepsilon\Lambda}$, so there is a tendency for ϕ to have the same sign at neighboring sites. In the language of statistical mechanics, the system is ferromagnetic. Also, only nearest neighbors are coupled. If a subset Π of Λ_ε divides Λ_ε into two disjoint subsets, and if X_1 and X_2 are random variables living on those subsets, then $E_{\varepsilon\Lambda} X_1 X_2 = E_{\varepsilon\Lambda} Y_1 Y_2$, where Y_i is the conditional expectation of X_i given the configuration on Π. This is the Markov property, which is exploited via the

Osterwalder-Schrader positivity property $E_{\varepsilon\Lambda}\theta X \cdot X \geqslant 0$ where θ is the reflection in a hyperplane Π.

Once again, quantum physics has turned into probability theory. The theory of random functions indexed by higher-dimensional spaces has been largely the province of those doing statistical mechanics. The imaginary time approach to constructive quantum field theory led to an extraordinarily fertile interaction of quantum physics and statistical mechanics.

The ultraviolet problem in two dimensions is relatively easy. Glimm and Jaffe do not give details in this book of their solution of the much harder ultraviolet problem in three dimensions. A variety of techniques from statistical mechanics is used to control the infrared limit, most notably correlation inequalities and cluster expansions. In Part I, among other things, the authors give beautiful expositions of these techniques in the simplest cases, and this eases the way for the quite difficult applications to field theory.

Once the Euclidean random field ϕ has been constructed, the corresponding quantum field may be obtained. There is one proviso: ϕ may not be ergodic under translations, which means that the quantum field may not have a unique vacuum. This is not a technicality. Indeed, ergodicity may fall, leading to a phase transition. The successes of constructive quantum field theory discussed here by Glimm and Jaffe have gone far beyond showing the existence of models—phase transitions, broken symmetry, particle structure, the scattering matrix, and other topics of physical interest have been thoroughly explored.

Functional integration has been far more successful in quantum physics than those of us who first learned the purely Hilbert-space approach ever dreamed. There is a mystery in this. Perhaps the mathematical trick of analytical continuation in time, which is applicable in some but not all situations, is not the key to the mystery. Perhaps probability theory has been so successful because the phenomena of quantum physics are inherently random phenomena. Whether this speculation is correct, only non-imaginary time will tell.

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Real elliptic curves, by Norman L. Alling, Mathematics Studies, Vol. 54, North-Holland Publishing Company, Amsterdam, 1981, xii + 350 pp., $36.25 US/Dfl. 85.00 paperback. ISBN 0-4448-6233-1

The author has (in collaboration with N. Greenleaf [**2**]) developed an interesting approach to real elliptic curves as an object of study in their own right, and not as a special case of complex analysis (as the universal imbedding subject). The theory was present in classical literature going back to 1882 (Klein [**8**]), and the historical context has stimulated the author to make a scholarly survey of elliptic functions from even before Gauss. This survey