the forcing **P** actually used, combines a condition s from **S** with an analogue of the set x from the Solovay conditions.

Obviously Solovay's trick can be iterated to code subsets of $\aleph_n$ by subsets of ω for any fixed n in ω. It is less obvious that this trick can be iterated infinitely many times in order to code, for example, a subset A of $\aleph_\omega$ as a subset of ω. In order to do this we must add, for each n in ω, a subset b_n of $\aleph_n$ which codes not only $A \cap \aleph_{n+1}$ but also the new subset b_{n+1} of $\aleph_{n+1}$ which we are simultaneously adding to code the segment of A above $\aleph_{n+1}$. This apparent infinite regress is avoided by the simple idea of allowing conditions for adding b_n to code up only that part of b_{n+1} which has already been forced.

If all cardinals were regular then this would essentially finish the argument: an Easton style class extension could be used to code a class A of ordinals. Unfortunately the basic Solovay forcing breaks down at singular cardinals: if κ is singular then trying to code a subset of κ^+ by a subset of κ will simply collapse κ. The reason for this is that the Solovay trick to code a subset of κ adds its subset of κ via conditions with domain of size less than κ. If κ is regular this is enough to give the conditions the κ chain condition and hence keep κ from being collapsed; with κ singular this fails. Jensen solves this difficulty in the coding problem by a use of the fine structure of L, and it is this use of fine structure which accounts for almost all the difficulty of the proof.

Since adding a subset b_ω of $\aleph_\omega$ to code $A \cap \aleph_{\omega+1}$ would collapse $\aleph_\omega$, Jensen makes the sequence $\langle b_n: n \in \omega \rangle$ code this up at the same time as each b_n is individually coding up $A \cap \aleph_{n+1}$. This alone seems difficult enough, but it must be recalled that we do not only have $\aleph_\omega$ to deal with; we must deal with all singular cardinals at once. Thus in constructing the new subset b_κ of κ we must keep track in some coherent way of all the singular cardinals λ larger than κ such that b_κ might be helping to code subsets of λ^+. This sort of organization is precisely what Jensen's principal $\square$ was designed for, and it is this use of the fine structure of L which leads to the complications of the proof.

This book gives a detailed proof which is relatively readable to anyone with the necessary prerequisites. Needless to say, these prerequisites include a firm grounding in the basic theory of fine structure as well as familiarity with set theory in general. There are numerous misprints, mainly in the most technical parts of the exposition, but these should not be too much of a barrier to the qualified reader. The exposition could also be improved by more explanation of where the proof is and where it is going, but the real difficulty of reading this book comes simply and directly from the difficulty of the mathematics.

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Random Fourier series with applications to harmonic analysis, by Michael B. Marcus and Gilles Pisier, Annals of Mathematics Studies, No. 101, Princeton Univ. Press, Princeton, N.J., 1981, v + 150 pp., $17.50 (cloth), $7.00 (paperback).

This little book is similar in spirit, as well as size and title, to Kahane's well-known monograph [**5**]. They both trace their origins back to a series of papers of Paley and Zygmund [**8**] which are now more than fifty years old. They both demonstrate that probabilistic methods can be of real value in studying problems in harmonic analysis which do not appear to be random and that harmonic analysis is a rich source of examples and questions for probabilists. One more similarity with Kahane's book is that the size is deceiving. It contains a great deal and requires, for those unfamiliar with the work of Dudley, Fernique and the authors, much additional reading.

Marcus and Pisier are concerned primarily with the question, raised initially by Paley and Zygmund, of the almost sure uniform convergence of the random Fourier series

$$\sum_{n=0}^{\infty} c_n \varepsilon_n e^{inx}, \qquad 0 \leq x \leq 2\pi, \tag{1}$$

where $\Sigma |c_n|^2 < \infty$ and $\{\varepsilon_n\}$ is the Rademacher sequence of random variables taking the values plus and minus one with equal probability. Here they give complete proofs of their necessary and sufficient condition which appear in [**6** and **7**].

In his monograph, Kahane states that he doesn't need much probability theory, that everything needed was well known by 1930. That is not the case here where recent work on the sample path properties of Gaussian processes is crucial. If the Rademacher functions in (1) are replaced by independent complex valued normal random variables then the series becomes a stationary complex valued Gaussian process. Dudley [**1**] studied the properties of the metric entropy of a natural norm associated with Gaussian processes and discovered a sufficient condition for such a process to have continuous sample paths almost surely. Fernique [**2**] showed that for stationary Gaussian processes the condition was necessary as well. The Dudley-Fernique theorem can be stated fairly simply for Fourier series. If

$$f(x) = \sum_{n=0}^{\infty} c_n e^{inx} \in L_2[0, 2\pi]$$

then there is a pseudo-metric d given by $d(s,t) = \| f_s - f_t \|_2$ where $f_t(x) = f(t + x)$. Let $N(\varepsilon)$ be the smallest number of ε balls which can be used to cover $[0, 2\pi]$. The entropy condition

$$\int_0^{\infty} [\log N(\varepsilon)]^{1/2}\, d\varepsilon < \infty$$

is necessary and sufficient for the almost sure continuity of the series $\Sigma c_n g_n e^{inx}$ where the g_n are independent Gaussian random variables.

The highlight of the book for many people will be the inclusion of Pisier's applications of random Fourier series to lacunary sets in harmonic analysis [**9**, **10**]. Here, moreover, they are presented in the non-Abelian case as well. Lacunary sets, such as Hadamard sets, Sidon sets and Λ_p sets, have long been of interest. A Sidon set of integers is a set E such that if

$$\sum_{n \in E} c_n e^{inx}$$

is continuous then $\sum |c_n| < \infty$; i.e., every continuous E function has an absolutely convergent Fourier series. There are many analytic characterizations of such sets going back to 1960 [**12**]. Since then the notions have been generalized and extended in every possible way. At the same time many simple but beautiful questions long remained unanswered. A good example is the question of whether the union of two Sidon sets is again a Sidon set, proved by Drury in 1970. In [**9**, **10**] Pisier answered another such question. If E is Sidon and f is an E function then it was shown [**12**] that $\| f \|_p \leqslant B\sqrt{p} \, \| f \|_2$ for some B and all $p < \infty$. Pisier showed the converse.

For a group G, the space Ca.s.(G) is the collection of L_2 functions f such that the randomized Fourier series f_ω of f almost surely represents a continuous function. It turns out that $\| f_\omega \|_\infty$ has finite expectation which is a norm that makes Ca.s.(G) into a Banach space. Now $A(G)$, the space of functions with absolutely convergent Fourier series, is the collection of functions such that

$$f = \sum h_n * k_n \tag{2}$$

where h_n and k_n are in L_2 and

$$\sum \| h_n \|_2 \| k_n \|_2 < \infty.$$

In [**9**] Pisier characterized Ca.s.(G) as the space of functions as in (2) where $h_n \in L_2$ and k_n is in the Orlicz space of class $L\sqrt{\log L}$. In addition there is a characterization of the dual space of Ca.s. in terms of multipliers. Combining this with the result of Rider [**11**] that C and Ca.s. have the same meaning for Sidon sets gives Pisier's results.

One nice feature of the book is the inclusion of random Fourier series on non-Abelian groups. Here the series are

$$\sum d_i \operatorname{Trace}\left[G_i U_i \hat{f}(i)\right]$$

where the U_i form a complete set of irreducible unitary representations of the group and $\hat{f}$ is the (matrix valued) Fourier transform. The random variables are matrix valued G_i. This section, while being quite similar to the commutative parts, is done in more detail. The lack of commutativity causes relatively few problems. The main one is the question of what are the differences between non-Abelian (p. 3, p. 74), non-commutative (p. 12) and noncommutative (p. 74).

The book should appeal greatly to probabilists who, as the authors hope, will be able to see random Fourier series as useful examples of Gaussian processes. It will naturally appeal to harmonic analysts who, however, will need to know some recent probability theory. From their point of view the book is not self-contained. It takes a lot of outside reading. (One good source is the paper of Jain and Marcus [**4**] for a survey of Gaussian processes including the Dudley-Fernique Theorem.) They might be disappointed that the book does not contain a proof of Pisier's result starting from scratch.

Many analysts would prefer using Rademacher functions to Gaussian random variables. They are so clear and simple, and one doesn't need to know a lot of probability to handle them. In the work of Hunt [3] and Kahane [5] it became clear that in many settings Rademacher functions are not the most convenient random variables to use. In this book the authors use Fernique's necessary condition for the continuity of stationary Gaussian processes. This depends on Slepian's lemma for which there is no version for non-Gaussian processes. This does not turn out to be a problem in the end since a Rademacher series is almost surely convergent if and only if the corresponding Gaussian series is. It does mean that it is necessary to use random variables other than Rademacher functions.

There might be some criticism that the book was written too early. The field is an active one and maybe the authors should have waited until more results were in and more applications to harmonic analysis could appear. (Pisier has recently obtained more characterizations of Sidon sets including arithmetic ones.) Maybe they should have taken more time and included a self-contained treatment of the probability. However, the book succeeds admirably in what it intends to do. It presents complete proofs of the authors' work announced in [7]. It includes non-Abelian groups with many simplifications of the original work. It brings a body of recent work in harmonic analysis and probability to a larger audience than the original papers. It will encourage probabilists to get involved in harmonic analysis and harmonic analysts to learn a great deal of interesting and important probability. That is, perhaps, the real value of such a book.

References

1. R. M. Dudley, *The sizes of compact subsets of Hilbert space and continuity of Gaussian processes*, J. Funct. Anal. **1** (1967), 290–330.

2. X. Fernique, *Des résultats nouveaux sur les processus gaussien*, C. R. Acad. Sci. Paris Sér. A **278** (1974), 363–365.

3. G. A. Hunt, *Random Fourier transforms*, Trans. Amer. Math. Soc. **71** (1951), 38–69.

4. N. C. Jain and M. B. Marcus, *Continuity of sub-gaussian processes*, Advances in Probability, vol. **4**, Dekker, New York, 1978.

5. J. P. Kahane, *Some random series of functions*, D.C. Heath, Lexington, Mass., 1968.

6. M. B. Marcus and G. Pisier, *Necessary and sufficient conditions for the uniform convergence of random trigonometric series*, Lecture Note Series, no. 50, Arhus University, Arhus, Denmark, 1978.

7. ______, *Random Fourier series on locally compact Abelian groups*, Séminarie de Probabilities XIII (Université de Strasbourg 1977/1978), Lecture Notes in Math., vol. 709, Springer-Verlag, Berlin and New York, 1979, pp. 72–89.

8. R. E. A. C. Paley and A. Zygmund, *On some series of functions*. (1), (2), (3), Proc. Cambridge Philos. Soc. **26** (1930), 337–357; **26** (1930), 458–474; **28** (1932), 190–205.

9. G. Pisier, *Sur l'espace de Banach des séries de Fourier aléatoires presque sûrement continues*, Exposé No. 17–18, Séminaire sur la géométrie des espace de Banach (1977–78), École Polytechnique, Palaiseau.

10. ______, *Ensembles de Sidon et processus gaussiens*, C. R. Acad. Sci. Paris Sér. A **286** (1978), 671–674.

11. D. Rider, *Randomly continuous functions and Sidon sets*, Duke Math. J. **42** (1975), 759–764.

12. W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227.

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