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A UNITED-SET FORMULA

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Let a morphism $f: X \to Y$ of algebraic varieties be given. A *united set* or *united k-tuple*[2] for f is a k-tuple $x_1, \ldots, x_k$ of distinct points on (or "infinitely near") X, such that $f(x_1) = \cdots = f(x_k)$. The purpose of this note is to announce an enumerative formula, valid under a restrictive hypothesis, for the united k-tuples of a map, i.e., a formula for the rational equivalence (or homology) class of a suitable cycle which parameterizes them. This yields as special cases formulas for the united k-tuples which contain a k_1-tuple, a k_2-tuple, etc. of mutually infinitely-near points. For our united-k-tuple cycle even to be defined, the morphism f has to admit a certain kind of "resolution" (essentially it must factor through a "generic" map into a variety fibred by smooth curves over Y). Our result is sufficient, however, to yield formulas for the lines having prescribed contacts with a given projective variety having "generic" singularities and arbitrary dimension and codimension; these in turn yield formulas for the Thom-Boardman-Roberts singularity schemes [**8**] of a generic projection of such a variety. Classically such formulas were known for curves, for surfaces in $\mathbf{P}^3$, and in a few other cases, cf. [**1**]. Some recent results were obtained by Lascoux [**6**], Roberts [**9**] and LeBarz [**7**]. Our result yields new formulas even for surfaces in $\mathbf{P}^4$. For a modern account of these and related matters, see Kleiman's surveys [**3**, **5**].

Admittedly, the hypothesis of existence of a "resolution" is a severe restriction on the morphism f. I am hopeful, however, that by pursuing further the same principles as in this paper, I will eventually obtain a united-set formula valid without such a restriction, and which would moreover be completely "intrinsic", in the sense of taking place on a suitable space associated solely to X (which is not the case with the present formula).

We shall work in the category of complete (usually nonsingular) varieties over a field. Everything goes through with no change, however, in the category of compact complex manifolds.

1. Set-up. Fix a morphism $f: X \to Y$ of nonsingular varieties, and put $m = \dim X$, $n = \dim Y$. A *resolution* of f is a diagram

$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

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[2]This term is *not* consistent with the classical one of united point of a correspondence.

where π is a smooth morphism of relative dimension 1, and $\tilde{f}$ is an embedding. More generally to define a *k-quasi-resolution* assume about f only that its "k'-fold locus" (cf. [4]), for all k', $2 \le k' \le k$, has codimension $> (k'-1)(n-m)$ in X (this is weaker than assuming that the latter locus has its expected codimension, which is $(k'-1)(n-m+1)$). Now fix such a k-quasi-resolution. We will define some auxiliary objects. Z^k is the k-fold fibre product of Z over Y; Z^k_X is the fibre product $Z^{k-1} \times_Y X$, with coordinate projections $\pi^k \colon Z^k_X \to Z^{k-1}$, $p^k \colon Z^k_X \to X$; $j^k \colon Z^k_X \to Z^k$ is $\pi^k \times \tilde{f}$; $\pi^k_i \colon Z^k_X \to Z$ is π^k followed by the ith coordinate projection; $q^k_i \colon Z^k_X \to Z^{k-1}_X$ is "delete the ith coordinate". For $i < j \le k$, the divisor $D^k_{i,j} \subset Z^k_X$ is defined by $\pi^k_i(\cdot) = \pi^k_j(\cdot)$; similarly, for $i < k$, $D^k_{i,k} \subset Z^k_X$ is defined by $\pi^k_i(\cdot) = p^k(\cdot)$; also put $D^k = \sum_{i=1}^{k-1} D^k_{i,k}$.

2. Formula. It is important to note that to each point $\mathbf{z} = (z_1, \ldots, z_{k-1}, x) \in Z^k_X$ there corresponds a well-defined *subscheme* $\sigma(\mathbf{z})$ of Z: namely this is the subscheme determined by the divisor $z_1 + \cdots + z_{k-1} + \tilde{f}(x)$ on the smooth curve $\pi^{-1}(\pi(z_1))$. Define the *united k-tuple locus* of f as

$$U_k(f) = \left\{ \mathbf{z} \in Z^k_X : \sigma(\mathbf{z}) \subset \tilde{f}(X) \text{ as schemes} \right\}.$$

This definition is justified by the fact that the image of $U_k(f)$ in the Hilbert scheme of X coincides, up to a lower-dimensional set, with the set of length-k subschemes of X which are mapped by f to a single reduced point. The image of $U_k(f)$ in X coincides with the "k-fold locus" of f as defined by Kleiman [4].

Now the set $U_k = U_k(f)$ can (see §4) naturally be made into a cycle of "expected dimension" $km-(k-1)n$ (i.e. expected codimension $(k-1)(n-m+1)$ in Z^k_X), and we seek a formula for the class $[U_k]$ of U_k in the Chow group of Z^k_X (though we could instead work in Z^k, working in Z^k_X yields finer results). The result we get is an inductive one, and goes as follows.

THEOREM. *Given a k-quasi-resolution as above, assume that U_k and U_{k-1} have their expected codimensions. Then if $k \ge 2$ we have*

$$\begin{aligned}[U_k] &= (\pi^k)^*(j^{k-1})_*([U_{k-1}]) \\ &\quad - \left(\sum_{i=1}^{k-1} (q^k_i)^*([U_{k-1}]) \cdot [D^k_{i,k}] \right) \left\{ \frac{(p^k)^*(c(\nu))}{1 + [D^k]} \right\}_{n-m}\end{aligned}$$

in $CH^{(k-1)(n-m+1)}(Z^k_X)$: where ν denotes the virtual normal bundle of $\tilde{f}$, i.e., $\tilde{f}^*(TZ) - TX$, $c(\nu)$ denotes its total Chern class, and $\{\ \}_{n-m}$ denotes the part in degree $n-m$. Also $U_1 = 1$.

3. Applications. By pushing the formula for $[U_k]$ down to X, we obtain a multiple-point formula à la Kleiman [4]. However, our formula contains more information than that. In particular, note that $U_k \cdot D^k_{i,j} \cdot D^k_{i',j'} \cdots$ parametrizes those united k-tuples whose ith and jth, i'th and j'th, etc. points are infinitely near each other, so the theorem yields enumerative formulas for the united k-tuples which are the union of Thom-Boardman $S_1^{(q_i)}$-singularities, $i = 1, \ldots, d$, $q_1 + \cdots + q_d = k$, cf. Roberts [8].

I know two main types of maps which admit quasi-resolutions.

(a) Let $g\colon V \to \mathbf{P}^N$ be a map whose image has generic singularities (cf. [3]). Put $X = \{(v,L) \in V \times G(1,\mathbf{P}^N) : g(v) \in L\}$, $Y = G(1,\mathbf{P}^N)$, and let $f\colon X \to Y$ be the projection, $Z \to Y$ the tautological $\mathbf{P}^1$ bundle, and $X \to Z$ the natural map. The united k-tuples of f correspond to the k-secant lines of $g(V)$, and thus the theorem yields enumerative formulas for these. They include formulas for the k-secant lines having prescribed types of contact with V, as well as for the "varieties of contact" of such lines. For instance, if V is a surface in $\mathbf{P}^4$, it will in general have a finite number of inflexional tangent lines meeting it elsewhere, say $L_1, \dots, L_r$; a formula for r was already given by LeBarz [7]. Put $L_i \cap V = 3p_i + q_i$. By pushing down to V the formula for $[U_4] \cdot [D^4_{1,2}] \cdot [D^4_{1,3}])$, we get a formula for the rational equivalence class of $p_1 + \dots + p_r$ (resp. $q_1 + \dots + q_r$) on V.

(b) Let $g\colon V \to \mathbf{P}^N$ be as above, and let $f\colon V \to Y = \mathbf{P}^n$ be g followed by projection from a general center $M = \mathbf{P}^{N-n-1} \subset \mathbf{P}^N$, where $n \geq \dim V$. Then projection from a general codimension-1 subspace $M' \subset M$ yields a quasi-resolution $\tilde{f}\colon V \to \mathbf{P}^{n+1}$, so the theorem applies, yielding some formulas for the singularities of f, including those of Thom-Boardman-Roberts as above. Actually this case is a special case of case (a), because united points of f correspond to k-secants of $\tilde{f}(V)$ passing through a fixed point.

4. Proof. As in other recent work on similar questions (see [3, 4, 5]), a key ingredient in the proof is an application of a "residual-intersection formula", of which we only require a relatively simple case, due to Fulton and MacPherson [2]. Consider the following cartesian diagram:

$$\begin{array}{ccc} I & \to & r^{-1}j^{k-1}U_{k-1} \\ \downarrow & & \downarrow \\ Z^k_X & \xrightarrow{j^k} & Z^k \end{array}$$

where $r\colon Z^k \to Z^{k-1}$ is projection onto the first $k-1$ coordinates. One can show that I consists of U_k plus a "residual" cycle, namely $(\pi^k)^{-1}(j^{k-1}U_{k-1}) \cdot D^k$. Now [2] tells us how to compute the contribution of this residual cycle to the intersection-cycle $[r^{-1}(j^{k-1}U_{k-1})] \cdot Z^k_X$, and this yields our formula.

ADDED IN PROOF. The hope expressed in the introduction is now a reality: a united-set formula taking place in a suitable "configuration space" $X^{[k]}$ and valid "modulo $\overline{S}_2(f)$" has been obtained, as a consequence of a general formula for the rational-equivalence class of $V^{[k]}$ on $Z^{[k]}$, where $V \subset Z$ are arbitrary manifolds. As an application, among others, I obtain a formula for the class, in the moduli space of curves, of the locus of curves carrying a g^r_d, for given r, d.

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