

BULLETIN (New Series) OF THE
 AMERICAN MATHEMATICAL SOCIETY
 Volume 8, Number 1, January 1983
 © 1983 American Mathematical Society
 0273-0979/82/0000-0782/\$01.75

Some applications of topological K-theory, by N. Mahammed, R. Piccinini and U. Suter, Mathematics Studies, no. 45, North-Holland, Amsterdam, ix + 137 pp., \$41.50.

1. K-theory. Topological K -theory was developed by Atiyah and Hirzebruch [4] by analogy with Grothendieck's K -groups in algebraic geometry. For a compact, Hausdorff space X one considers vector bundles, $\pi: E \rightarrow X$. That is, E is locally a product of a neighborhood U in X with a (complex, let us say) vector space so that π is locally a projection onto U . Take isomorphism classes of such (complex) vector bundles and form the group generated by these subject to the condition that the fibre-by-fibre direct sum $E \oplus F$ shall represent the group-sum of the classes of E and F . The resulting group is $K(X)$.

$K(X)$ has been called "the linear algebra of algebraic topology". As we have seen with direct sums, natural vector space operations can be performed fibre-by-fibre on vector bundles. For example, tensor product gives $K(X)$ a ring structure while the exterior power constructions, $\lambda^k E$, induce natural operations on $K(X)$ from which further useful operations can be constructed—the most celebrated of these are the Adams operations, ψ^k , which are ring homomorphisms. A map $f: Y \rightarrow X$ enables us to construct a vector bundle f^*E over Y , given by $f^*E = \{(e, y) \in E \times Y \mid \pi(e) = f(y)\}$. In fact $f^*: K(X) \rightarrow K(Y)$ is a ring homomorphism and $\psi^k f^*(z) = f^*(\psi^k(z))$ —a simple property from which many famous results in algebraic topology have been deduced.

One may construct a homomorphism $\varepsilon: K(X) \rightarrow Z$ by choosing $x_0 \in X$ and assigning to the class of E the dimension of $\pi^{-1}(x_0)$. This homomorphism is split, by sending n to $X \times \mathbf{C}^n$, and we may set $\ker \varepsilon = \tilde{K}(X)$. Bott [6] proved that

$$\tilde{K}(S^n) = \begin{cases} 0 & \text{if } n \text{ odd,} \\ Z & \text{if } n \text{ even,} \end{cases}$$

thereby opening the way to calculations of $K(X)$ and thereafter to applications to algebraic topology. For example, let P^n denote the projection space obtained from the n -sphere, S^n , by identifying antipodal points. $\tilde{K}(P^n)$ provides a fine example of the "linear algebra" aspects of K -theory. A vector bundle, the Hopf bundle, H , is formed by taking $S^n \times \mathbf{C}$ and identifying (z, v) with $(-z, -v)$. In fact $H - 1$ generates $\tilde{K}(P^n)$, which is a cyclic group of 2-primary order. If one has a nondegenerate bilinear map $\phi: \mathbf{R}^{n+1} \times \mathbf{C}^m \rightarrow \mathbf{C}^m$ (for example: \mathbf{R}^m might be a module over the Clifford algebra $C(\mathbf{R}^{n+1})$) then $0 = m(H - 1) \in \tilde{K}(P^n)$ by virtue of the isomorphism $S^n \times \mathbf{C}^m \rightarrow S^n \times \mathbf{C}^m$ which sends (z, v) to $(z, \phi(z, v))$ and induces an isomorphism $mH \cong P^n \times \mathbf{C}^m$. A Clifford module structure on \mathbf{R}^m , such as ϕ above, can be used to transport, in a continuous manner, a copy of \mathbf{R}^{n+1} tangent to a chosen point on S^{m-1} in such a way as to give a subvector bundle of the form $S^{m-1} \times \mathbf{R}^{n+1}$ of the tangent vector bundle of S^{m-1} , $\tau_{S^{m-1}}$. Adams [1] showed that this construction

produces the maximal-dimensional product subvector bundle of $\tau_{S^{m-1}}$ and his proof, as one might expect from the above, uses the structure of the groups $\tilde{K}(P^n)$ and the action thereon of the ψ^k -operations.

In 1931 Hopf [7] constructed a continuous map $f: S^3 \rightarrow S^2$ which could not be continuously deformed into a constant map. Hopf detected the nontriviality of f by means of a homomorphism $H: \pi_{2n+1}(S^{n+1}) \rightarrow Z$, called the Hopf invariant. When $n = 1$ the Hopf map defined a homotopy class in the homotopy group $\pi_3(S^2)$ with $H(f) = \pm 1$. From a map $\mu: S^n \times S^n \rightarrow S^n$ (which might, for example, come from the normalised restriction of a nondegenerate, bilinear map $\phi: \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \rightarrow \mathbf{R}^{n+1}$) the Hopf construction produces a map $f(\mu): S^{2n+1} \rightarrow S^{n+1}$. The celebrated ‘‘Hopf invariant one’’ problem consisted in determining the possibilities for n in order that $H(f(\mu)) = \pm 1$. The complex, quaternionic and Cayley multiplications (on \mathbf{R}^2 , \mathbf{R}^4 , \mathbf{R}^8 respectively) give three examples. $K(X)$ may be used to show that $n = 1, 3, 7$ are the only possibilities. In fact, in order to lighten the lump for readers familiar with algebraic topology here is a sketch of one such proof which is probably unfamiliar even to the experts. Firstly one chooses one’s favourite reformulation of the problem. For us this will be the following. We have to show (in the case when $n + 1 = 2t$) that there is no map $\lambda: S^{2t+N-1} \rightarrow \Sigma^N P^{2t-1}$ such that $\gamma\lambda$ has degree one. Here the map, γ , collapses $\Sigma^N P^{2t-2}$ to a point, to give S^{2t+N-1} and by $\Sigma^N X$ we mean the N -fold suspension of X given by $X \times D^N / (X \times S^{N-1} \cup x_0 \times D^N)$. The K -homology group $K_1(\Sigma^N P^{2t-1})$ is defined to be $\tilde{K}(Y)$ for Y a complement of $\Sigma^N P^{2t-1}$ when the latter is embedded in a suitably high-dimensional sphere. For our purposes we need to know only that

(i)

$$\text{if } N = 2M, K_1(\Sigma^N P^{2t-1}) \cong Z\langle F_2 \rangle \oplus Z/2^{t-1}\langle F_1 \rangle$$

and

(ii)

$$\text{The existence of } \lambda \text{ implies } \psi^3(F_2) = 3^{M+t}F_2.$$

However, in fact, $\psi^3(F_2) = 3^M(((3^t - 1)/2)F_1 + 3^t F_2)$ so that t must satisfy $3^t \equiv 1 \pmod{2^t}$ in order for λ to exist; so that $t = 1, 2$ or 4 .

2. What about the book, eh? At the risk of becoming too technical I will now tell you what the book contains—and why. It attempts, quite successfully, to chart results and progress in several topics in algebraic topology which may be analysed with the aid of K -theory calculations. The book assumes a basic knowledge of K -theory and does not attempt to be as comprehensive in coverage of basic properties as either M. Karoubi [8] or M. F. Atiyah [2]. However, the authors guide the reader adequately to the sources, among their two hundred and seventy references, whence basic facts may be sought.

Each topic is introduced with a motivating discussion and an extensive sketch of its historical background. This, I imagine, will appeal to those who are beginning to learn about K -theory and have intentions of doing some serious work on it.

The first topic to be treated is the solution of the Hopf invariant one problem. Although covered just as well in many other places one can hardly, I suppose, expect the authors to refrain from taking delight in the history and details of the Hopf construction, nonexistence of real division algebras and other classical paraphernalia of this topic. After all, I could not resist the temptation myself! In any case, a great deal of the Hopf invariant material serves as an easy first approximation to the chapter in which they tackle classification of rank two, torsion free, finite H -spaces.

The H -space chapter was the one I enjoyed most. Here the authors study the K -theory of the projective plane of such an H -space and obtain a classification of the possible torsion free, rank two cohomology types. In addition they give a K -theoretic proof of Zabrodsky's result concerning which cell complexes, $S^3 \cup_{n\omega} e^7 \cup e^{10}$, are H -spaces and finally a homotopy classification of torsion free, rank two, finite H -spaces.

Almost complex and stably complex structures on manifolds are introduced with a sketch of their historical development. The authors give an example, due to W. A. Sutherland, which illustrates that the existence of a stably complex structure is not solely a property of the homotopy type of the manifold.

A chapter is devoted to the solution of the vector field problem for S^n and at the same time the analogous problems are solved for complex and quaternionic vector fields on $S(\mathbf{F}^n)$ ($\mathbf{F} = \mathbf{C}$ or \mathbf{H} respectively). This chapter sets up the notions of J -equivalence, (co)-reducibility and introduces the groups, $J(X)$. In a further chapter the equivariant analogues of all these notions are set up to study the vector field problem for linear spherical space forms along the lines of J. C. Becker [5].

There is a chapter on immersions and embedding of manifolds which covers the background and history briskly. Calculations, using a method of M. F. Atiyah [3], are used to give nonimmersion results for Dold manifolds, projective spaces and some spherical space forms.

There is a minute chapter on maps between classifying spaces. This fits in very nicely with the calculations developed earlier in the book. However, this section could do with some more meat. For example, here would have been an ideal place for the simple K -theory calculation (which I learned from J. F. Adams) which determines the possible degrees of self maps of $\mathbf{H}P^\infty$. In fact, they might have gone further and surveyed some of J. F. Adams' work on "Maps between classifying spaces" along with applications such as the nice proof of C. B. Thomas [9] that the Smale conjecture implies $Q(8n, k, l)$ cannot act freely on S^3 . In point of fact, neither of the sources are even mentioned in the references.

The book ends with a chapter which gives an outline of the Atiyah-Singer Index Theorem and its corollaries. This is obviously a very worthwhile topic to include in a K -theory book but unfortunately one chapter does not do justice to it and here, again, I noticed many applications which were missing from the bibliography.

All in all, however, I think the authors have done a reasonable job in producing a timely volume which fills a real gap in the expository research literature.

REFERENCES

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632.
2. M. F. Atiyah, *K-theory*, Benjamin, New York, 1967.
3. _____, Topology **1** (1961), 125–132.
4. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R.I., 1961, pp. 7–38.
5. J. C. Becker, *The span of spherical space forms*, Amer. J. Math. **94** (1972), 991–1025.
6. R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **75** (1953), 409–448.
7. H. Hopf, *Über die abbildungen der dreidimensionalen sphäre auf die Kugelfläche*, Math. Ann. **104** (1931), 637–665.
8. M. Karoubi, *K-theory*, Grundlehren der. Math. Wiss., no. 226, Springer-Verlag, Berlin and New York, 1978.
9. C. B. Thomas, *Free actions by finite groups on S^3* , Proc. Sympos. Pure Math., vol. 32, Amer. Math. Soc., Providence, R.I., 1978, pp. 125–130.

VICTOR SNAITH

BULLETIN (New Series) OF THE
 AMERICAN MATHEMATICAL SOCIETY
 Volume 8, Number 1, January 1983
 © 1983 American Mathematical Society
 0273-0979/82/0000-0779/\$03.50

Finite simple groups, an introduction to their classification, by Daniel Gorenstein, Plenum Publishing Corporation, New York, 1982, x + 333 pp., \$29.50.

After an effort of about 30 years involving more than a hundred mathematicians in several countries, the finite simple groups have apparently been classified. I use the work “apparently” because this achievement is a unique sociological phenomenon in the history of mathematics. Proofs of the many relevant results are scattered throughout the mathematical literature in 300–500 papers covering 5,000–10,000 pages. No individual has gone through the whole proof and checked all the details. This is not an entirely satisfactory situation. However most experts are convinced that the proof is essentially correct; any errors which occur are expected to be minor oversights or local errors which can be corrected by the methods that have been developed in the process of completing the classification. More importantly, no error is expected to change the end result, i.e. to lead to new simple groups.

Except at the level of foundations, mathematics is not a matter of faith, so it is not surprising that the announcement of the classification has been treated with some scepticism among mathematicians. Nevertheless it would be pointless (and probably impossible) for anyone at present to attempt to go through a complete proof and to check all the details, because the proof is continually being revised, simplified and shortened. This process has been dubbed revisionism. In part, such simplifications are due to the inevitable redundancy which occurs in an undertaking of this magnitude. However great successes have already been achieved for less obvious reasons. For example, the classification of simple groups with dihedral Sylow 2-groups originally took 221 journal pages [7, 8], there is now a proof in 29 pages [1, 2]. Such a simplification and others of a similar sort are due to the introduction of new ideas and to a deeper understanding of the structure of finite simple groups. In these circumstances brevity and clarity go hand in hand. It is not unreasonable to